Economics 212A, Homework 3 Anton Cheremukhin

December 16, 2006

Exercise 1

a) Bellman equations for the two states:
$rV_0 = 0 + \delta\left(E_{\theta'} \max_{\alpha_0(\theta')\in\{0,1\}}\left[\alpha_0(\theta')\left[V_1(\theta') - p\right] + (1-\alpha_0(\theta'))V_0\right] - V_0\right)$
$rV_1(\theta) = \theta + \delta\left(E_{\theta'} \max_{\alpha_1(\theta')\in\{0,1\}}\left[\alpha_1(\theta')V_1(\theta') + (1-\alpha_1(\theta'))(V_0+p)\right] - V_1(\theta)\right)$
$\theta' \sim f(\theta)$ is iid, defined on $\left[\underline{\theta},\overline{\theta}\right]$
b) $\alpha_0(\theta') = 1$ iff $V_1(\theta') - p \geq V_0$ $\qquad \alpha_1(\theta') = 1$ iff $V_1(\theta') \geq V_0 + p$
Therefore, $\alpha_0(\theta') = \alpha_1(\theta')$ except for a set of measure zero.
$V_1(\theta)$ is strictly increasing in θ because the Bellman equation preserves this property, and defines a contraction mapping ($\frac{\delta}{r+\delta} < 1$). I.e. if we take an initial guess of $V_1(\theta)$ that is strictly increasing in θ, then every next iteration of the value function iteration procedure will have this property, and since it is a contraction mapping, the fixed point will also have this property. Therefore, the equation $V_1(\theta') = V_0 + p$ has a unique solution, defining a cutoff rule θ^*: $\qquad \alpha_i(\theta) = \begin{cases} 1 & \theta \geq \theta^* \\ 0 & \theta < \theta^* \end{cases}$

c) A steady-state equilibrium is a collection of objects: $\{V_0, V_1(\theta), \theta^*, p\}$
which satisfy the Bellman equations:

$$(1)\ \frac{r+\delta}{\delta}V_0 = \int_{\underline{\theta}}^{\theta^*} V_0 f(\theta')\,d\theta' + \int_{\theta^*}^{\overline{\theta}} \left[V_1(\theta') - p\right] f(\theta')\,d\theta'$$

$$(2)\ \frac{r+\delta}{\delta}V_1(\theta) = \frac{\theta}{\delta} + \int_{\underline{\theta}}^{\theta^*} (V_0+p) f(\theta')\,d\theta' + \int_{\theta^*}^{\overline{\theta}} \left[V_1(\theta')\right] f(\theta')\,d\theta'$$

the supply equation: $\qquad (3)\ S = E_{\theta'}\alpha(\theta') = \int_{\theta^*}^{\overline{\theta}} f(\theta')\,d\theta'$

and the cutoff equation: (4) $V_1(\theta^*) = V_0 + p$
From the Bellman equations if follows that

$$\frac{r+\delta}{\delta}V_0 + p\int_{\theta^*}^{\overline{\theta}} f(\theta')\,d\theta' = \int_{\underline{\theta}}^{\theta^*} V_0 f(\theta')\,d\theta' + \int_{\theta^*}^{\overline{\theta}} V_1(\theta') f(\theta')\,d\theta' = \frac{r+\delta}{\delta}V_1(\theta) - \frac{\theta}{\delta} - p\int_{\underline{\theta}}^{\theta^*} f(\theta')\,d\theta'$$

Therefore, $\frac{r+\delta}{\delta}V_0 + p = \frac{r+\delta}{\delta}V_1(\theta) - \frac{\theta}{\delta}$ for any θ. $\qquad$ Besides, $V_1(\theta^*) = V_0 + p$
$\Rightarrow \qquad \frac{r+\delta}{\delta}V_0 + p = \frac{r+\delta}{\delta}V_1(\theta^*) - \frac{\theta^*}{\delta} = \frac{r+\delta}{\delta}(V_0+p) - \frac{\theta^*}{\delta}$ $\qquad$ Hence, $p = \frac{\theta^*}{r}$.

d) The number of sell transactions is equal to the number of agents transiting from $\theta \geq \theta^*$ to $\theta' \leq \theta^*$. Both are iid, so it is equal to the product of the probabilities of being above and being below. $N_s = \Pr[\theta \geq \theta^*]\Pr[\theta \leq \theta^*] = \Pr[\theta \geq \theta^*](1 - \Pr[\theta \geq \theta^*])$

Notice, that $\Pr[\theta \geq \theta^*] = \int_{\theta^*}^{\overline{\theta}} f(\theta')\,d\theta' = S$. Hence, N_s=S$(1-S)$.

The total number of transactions per unit of time is $\delta(N_s + N_b) = 2\delta S(1-S)$.

Exercise 2

a) Let there be a cost c of each transaction, denote $p_a = p + c$, $p_b = p - c$.
$rV_0 = 0 + \delta\left(E_{\theta'} \max_{\alpha_0(\theta')\in\{0,1\}}\left[\alpha_0(\theta')\left[V_1(\theta') - p_a\right] + (1-\alpha_0(\theta'))V_0\right] - V_0\right)$
$rV_1(\theta) = \theta + \delta\left(E_{\theta'} \max_{\alpha_1(\theta')\in\{0,1\}}\left[\alpha_1(\theta')V_1(\theta') + (1-\alpha_1(\theta'))(V_0+p_b)\right] - V_1(\theta)\right)$
b) $\alpha_0(\theta') = 1$ iff $V_1(\theta') - p_a \geq V_0$ $\quad$ $\alpha_1(\theta') = 1$ iff $V_1(\theta') \geq V_0 + p_b$

By similar logic $V_1(\theta)$ is strictly increasing in θ because the Bellman equation preserves this property, and defines a contraction mapping ($\frac{\delta}{r+\delta} < 1$). Then the equation $V_1(\theta_i) = V_0 + p_i$ has a unique solution, defining a cutoff rule θ_i^* (i=a,b): $\quad \alpha_i(\theta) = \begin{cases} 1 & \theta \geq \theta_i^* \\ 0 & \theta < \theta_i^* \end{cases}$.

Also, notice that $p_a > p_b \Rightarrow \theta_a > \theta_b$. Therefore, when $\theta' < \theta_b$ people only sell, when $\theta' > \theta_a$ people only buy, when $\theta_b < \theta' < \theta_a$ people neither buy nor sell.

c) So, agents with $\theta < \theta_b$ hold zero units of asset, agents with $\theta > \theta_a$ hold one unit of asset. Denote the probability of getting a low $\theta < \theta_b$ by $q_L = \int_{\underline{\theta}}^{\theta_b^*} f(\theta')\, d\theta'$and the probability of getting a high $\theta > \theta_a$ by $q_H = \int_{\theta_a^*}^{\overline{\theta}} f(\theta')\, d\theta'$. Out of all agents the ones coming to the middle state from high state are of measure $q_H(1 - q_H - q_L)$ and the ones coming to the middle state from low state are of measure $q_L(1 - q_H - q_L)$. By Bayes rule, the fraction $\frac{q_H}{q_H+q_L}$ of agents in the middle state will hold one unit of asset, and the fraction $\frac{q_L}{q_H+q_L}$ of agents in the middle state will hold zero units of asset.

d) A steady-state equilibrium is a collection of objects: $\{V_0, V_1(\theta), \theta_a^*, \theta_b^*, p_a, p_b\}$
which satisfy the Bellman equations:

(1) $\frac{r+\delta}{\delta}V_0 = \int_{\underline{\theta}}^{\theta_a^*} V_0 f(\theta')\, d\theta' + \int_{\theta_a^*}^{\overline{\theta}} \left[V_1(\theta') - p_a\right] f(\theta')\, d\theta'$

(2) $\frac{r+\delta}{\delta}V_1(\theta) = \frac{\theta}{\delta} + \int_{\underline{\theta}}^{\theta_b^*} (V_0 + p_b) f(\theta')\, d\theta' + \int_{\theta_b^*}^{\overline{\theta}} \left[V_1(\theta')\right] f(\theta')\, d\theta'$

Supply equation: (3) $S = E_{\theta'}\alpha(\theta') = \dfrac{\int_{\theta_a^*}^{\overline{\theta}} f(\theta')d\theta'}{1-\int_{\theta_b^*}^{\theta_a^*} f(\theta')d\theta'} \int_{\theta_b^*}^{\theta_a^*} f(\theta')\, d\theta' + \int_{\theta_a^*}^{\overline{\theta}} f(\theta')\, d\theta' = \dfrac{\int_{\theta_a^*}^{\overline{\theta}} f(\theta')d\theta'}{1-\int_{\theta_b^*}^{\theta_a^*} f(\theta')d\theta'} = \frac{q_H}{q_L+q_H}$

and the cutoff equations: (4-6) $V_1(\theta_a^*) = V_0 + p_a$ $\quad$ $V_1(\theta_b^*) = V_0 + p_b$ $\quad$ $p_a - p_b = 2c$.

e) $V_1(\theta_a) = V_0 + p_a$ $\quad$ $V_1(\theta_b) = V_0 + p_b$ $\quad$ $p_a - p_b = 2c$

Combining the two Bellmans we get: $\frac{\theta}{\delta} + p_a q_H + p_b q_L = \frac{r+\delta}{\delta}(V_1(\theta) - V_0) - \int_{\theta_b}^{\theta_a} (V_1(\theta) - V_0) f(\theta)\, d\theta$

$\frac{\theta}{\delta} + (V_1(\theta_a) - V_0)q_H + (V_1(\theta_b) - V_0)q_L = \frac{r+\delta}{\delta}(V_1(\theta) - V_0) - \int_{\theta_b}^{\theta_a} (V_1(\theta) - V_0) f(\theta)\, d\theta$

$$V_1(\theta) - V_0 = \frac{\theta}{\delta+r} + \frac{\delta}{r+\delta}\left[(V_1(\theta_a) - V_0)\, q_H + (V_1(\theta_b) - V_0)\, q_L + \int_{\theta_b}^{\theta_a} (V_1(\theta) - V_0)\, f(\theta)\, d\theta\right] =$$

$$= \frac{\theta}{r+\delta} + \frac{\delta}{r+\delta}A \qquad A = \left(\frac{\theta_a}{\delta+r} + \frac{\delta}{r+\delta}A\right) q_H + \left(\frac{\theta_b}{\delta+r} + \frac{\delta}{r+\delta}A\right) q_L + \frac{1}{\delta+r}\int_{\theta_b}^{\theta_a} \theta f(\theta)\, d\theta + \frac{\delta}{r+\delta}A\,(1 - q_H - q_L)$$

$$A = \frac{\theta_a}{r} q_H + \frac{\theta_b}{r} q_L + \frac{1}{r}\int_{\theta_b}^{\theta_a} \theta f(\theta)\, d\theta \qquad V_1(\theta) - V_0 = \frac{\theta}{r+\delta} + \frac{\delta}{r}\frac{\theta_a}{r+\delta} q_H + \frac{\delta}{r}\frac{\theta_b}{r+\delta} q_L + \frac{\delta}{r}\frac{1}{r+\delta}\int_{\theta_b}^{\theta_a} \theta f(\theta)\, d\theta$$

$$p_a = V_1(\theta_a) - V_0 = \frac{\theta_a}{r+\delta} + \frac{\delta}{r}\frac{\theta_a}{r+\delta} q_H + \frac{\delta}{r}\frac{\theta_b}{r+\delta} q_L + \frac{\delta}{r}\frac{1}{r+\delta}\int_{\theta_b}^{\theta_a} \theta f(\theta)\, d\theta = p + c$$

$$p_b = V_1(\theta_b) - V_0 = \frac{\theta_b}{r+\delta} + \frac{\delta}{r}\frac{\theta_a}{r+\delta} q_H + \frac{\delta}{r}\frac{\theta_b}{r+\delta} q_L + \frac{\delta}{r}\frac{1}{r+\delta}\int_{\theta_b}^{\theta_a} \theta f(\theta)\, d\theta = p - c$$

Hence, $\frac{\theta_a}{\delta} - \frac{\theta_b}{\delta} = \frac{r+\delta}{\delta}\left(V_1(\theta_a) - V_1(\theta_b)\right) = \frac{r+\delta}{\delta}(p_a - p_b) = \frac{r+\delta}{\delta}2c \qquad \Rightarrow \qquad \boxed{\frac{\theta_a - \theta_b}{r+\delta} = 2c}$

$$\boxed{p = \frac{\theta_a + \theta_b}{2(r+\delta)} + \frac{\delta}{r}\frac{\theta_a}{r+\delta} q_H + \frac{\delta}{r}\frac{\theta_b}{r+\delta} q_L + \frac{\delta}{r}\frac{1}{r+\delta}\int_{\theta_b}^{\theta_a} \theta f(\theta)\, d\theta} = \frac{\theta_a}{r+\delta} - \frac{1}{2}\frac{\theta_a - \theta_b}{r+\delta} + \frac{\delta}{r+\delta}\frac{\theta_a}{r} q_H + \frac{\delta}{r+\delta}\frac{\theta_b}{r} q_L + \frac{1}{r}\frac{\delta}{r+\delta}\int_{\theta_b}^{\theta_a} \theta f(\theta)\, d\theta$$

Since $\frac{\delta}{r+\delta}\frac{\theta_b}{r} q_L + \frac{2}{r}\frac{\delta}{r+\delta}\int_{\theta_b}^{\theta_a} \theta f(\theta)\, d\theta = \frac{\delta}{r+\delta}\frac{\theta_a}{r}(1 - q_H) \qquad \Leftrightarrow \qquad \theta_b q_L + 2\int_{\theta_b}^{\theta_a} \theta f(\theta)\, d\theta = \theta_a (1 - q_H)$

$$\boxed{p} \neq \boxed{\frac{\theta_a}{r} - \frac{\theta_a - \theta_b}{2(r+\delta)} - \frac{1}{r}\frac{\delta}{r+\delta}\int_{\theta_b}^{\theta_a} \theta f(\theta)\, d\theta} \qquad \text{unless} \quad \int_{\theta_b}^{\theta_a} 2\theta f(\theta)\, d\theta = \int_{\underline{\theta}}^{\theta_a} \theta_a f(\theta)\, d\theta - \int_{\underline{\theta}}^{\theta_b} \theta_b f(\theta)\, d\theta$$

i.e. this formula would only be true if the distribution is uniform: $f(\theta) = 1, \underline{\theta} = 0$

f) $S = \frac{q_H}{q_L + q_H} \Rightarrow \qquad (q_L + q_H)\, S = q_H \Rightarrow \qquad (1 - S)\, q_H = S q_L \Rightarrow \qquad -\delta(1 - S)\, q_H + \delta S q_L = 0$

$$q_H = \int_{\theta_a}^{\overline{\theta}} f(\theta')\, d\theta' = 1 - \int_{\underline{\theta}}^{\theta_a} f(\theta')\, d\theta' = 1 - F(\theta_a) \qquad q_L = F(\theta_b) = F(\theta_a - 2c(r+\delta))$$

Hence, $\boxed{-\delta(1 - S)(1 - F(\theta_a)) + \delta S F(\theta_a - 2c(r+\delta)) = 0}$

The LHS is the flow of demand (buy orders) for the asset, and the RHS is the flow of supply (sell orders) for the asset. Demand equals supply. The function is strictly increasing in θ_a, so the solution is unique.

g) Assuming the statement (2) in (e) was right $p = \frac{\theta_a}{r} - \frac{\theta_a - \theta_b}{2(r+\delta)} - \frac{1}{r}\frac{\delta}{r+\delta}\int_{\theta_b}^{\theta_a} \theta f(\theta)\, d\theta \qquad \frac{\theta_a - \theta_b}{2(r+\delta)} = c$

$$-\delta(1 - S)(1 - F(\theta_a)) + \delta S F(\theta_a - 2c(r+\delta)) = 0$$

$$\frac{\partial\theta_a}{\partial c} - \frac{\partial\theta_b}{\partial c} = 2(r+\delta) \qquad \delta(1 - S)\, f(\theta)\frac{\partial\theta_a}{\partial c} + \delta S f(\theta)\left(\frac{\partial\theta_a}{\partial c} - 2(r+\delta)\right) = 0$$

Therefore, $\frac{\partial\theta_a}{\partial c} = 2S(r+\delta) \qquad \frac{\partial\theta_b}{\partial c} = \frac{\partial\theta_a}{\partial c} - 2(r+\delta) = 2(r+\delta)(S - 1)$

Hence, $\boxed{\frac{\partial p}{\partial c}} = \frac{1}{r}\frac{\partial\theta_a}{\partial c} - 1 - \frac{1}{r}\frac{\delta}{r+\delta}\theta f(\theta)\left(\frac{\partial\theta_a}{\partial c} - \frac{\partial\theta_b}{\partial c}\right) = \boxed{2S\frac{r+\delta}{r} - 1 - \frac{2\delta}{r}\theta f(\theta)}$

In general, however, $p = \frac{\theta_a+\theta_b}{2(r+\delta)} + \frac{\delta}{r}\frac{\theta_a}{r+\delta}q_H + \frac{\delta}{r}\frac{\theta_b}{r+\delta}q_L + \frac{\delta}{r}\frac{1}{r+\delta}\int_{\theta_b}^{\theta_a}\theta f(\theta)\,d\theta$

$\boxed{\frac{\partial p}{\partial c}} = \frac{1}{2(r+\delta)}\left(\frac{\partial\theta_a}{\partial c} + \frac{\partial\theta_b}{\partial c}\right) + \frac{\delta}{r}\frac{1}{r+\delta}\left(q_H\frac{\partial\theta_a}{\partial c} + \frac{\partial\theta_b}{\partial c}q_L\right) + \frac{\delta}{r}\frac{1}{r+\delta}\left(\theta_b f(\theta_b) - \theta_a f(\theta_a)\right) +$
$+ \frac{1}{r}\frac{\delta}{r+\delta}\theta f(\theta)\left(\frac{\partial\theta_a}{\partial c} - \frac{\partial\theta_b}{\partial c}\right) =$
$= \boxed{(2S-1) + 2\frac{\delta}{r}\left(Sq_H + (S-1)q_L\right) + \frac{2\delta}{r}\theta f(\theta)}$

g) $f(\theta) = 1 \qquad F(\theta) = \theta \qquad \theta \in [0,1]$
$(1-S)(1-\theta_a) = S(\theta_a - 2c(r+\delta)) \qquad \Rightarrow \qquad \theta_a = 1 - S + 2cS(r+\delta) = 1 - S + S(\theta_a - \theta_b)$
$\theta_a = 1 - \frac{S}{1-S}\theta_b$ When $c \to 0,\ \theta = 1 - S.$
Hence, $\frac{\partial p}{\partial c} = 2S\frac{r+\delta}{r} - 1 - \frac{2\delta}{r}(1-S) = \frac{1}{r}(2S-1)(r+2\delta) > 0$ if $\boxed{S > \frac{1}{2}}$.

Exercise 3

a) If $2K \geq 2\delta S(1-S)$ then the competitive marketmakers can hold the whole capacity, so the price of transactions is zero. There is no transaction cost. In this case there is only one cut-off as in question 2, $p = \theta^*/r$.

b) If $2K < 2\delta S(1-S)$, then the competitive marketmakers can hold the whole capacity, so the price of transactions has to go up. The number of selling orders in this case is equal to the number of buying orders and equal to $\frac{q_H q_L}{q_L+q_H}$. The number of transactions per unit of time is $2\delta\frac{q_H q_L}{q_L+q_H}$. In equilibirum it must be equal to capacity: $2\delta\frac{q_H q_L}{q_H+q_L} = 2K$.

$q_L = F(\theta_b) \qquad q_H = 1 - F(\theta_a) \qquad (1-S)(1-F(\theta_a)) = SF(\theta_b)$
Hence, $\delta SF(\theta_b) = K \qquad \Rightarrow \qquad F(\theta_b) = \frac{1}{S}\frac{K}{\delta} \qquad F(\theta_a) = 1 - \frac{S}{1-S}F(\theta_b) = 1 - \frac{1}{1-S}\frac{K}{\delta}$.

c) The bid-ask spread is $\boxed{p_a - p_b} = 2c = \frac{\theta_a - \theta_b}{(r+\delta)} = \boxed{\frac{F^{-1}\left(1-\frac{1}{1-S}\frac{K}{\delta}\right) - F^{-1}\left(\frac{1}{S}\frac{K}{\delta}\right)}{r+\delta}}$.

Since $F(.)$ is an increasing function, so is $F^{-1}(.)$. Hence, the bid-ask spread decreases when capacity goes up.

When δ increases $F^{-1}\left(1-\frac{1}{1-S}\frac{K}{\delta}\right) - F^{-1}\left(\frac{1}{S}\frac{K}{\delta}\right)$ increases, but $r+\delta$ also increases. So the effect of the intensity of shock on the bid-ask spread is indeterminate. On one hand the fraction of the population, which wants to buy and sell, but cannot do it increases. On the other hand, the shocks come more often, and the time of match goes down, so the value of a match is discounted at a higher rate.