Economics 203C, Homework #2 Anton Cheremukhin

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Exercise 1

a) $A = \begin{bmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{1l} \\ b_{k1} & b_{kl} \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{1q} \\ c_{p1} & c_{pq} \end{bmatrix}, D = \begin{bmatrix} d_{11} & d_{1s} \\ d_{r1} & d_{rs} \end{bmatrix}$

a1) $(A \otimes B)(C \otimes D) =$ (m*k)x(n*l) (p*r)x(q*s) Need $n = p, l = r$

$$\begin{bmatrix} a_{11}b_{11} & a_{11}b_{1l} & a_{1n}b_{11} & a_{1n}b_{1l} \\ a_{11}b_{k1} & a_{11}b_{kl} & a_{1n}b_{k1} & a_{1n}b_{kl} \\ a_{m1}b_{11} & a_{m1}b_{1l} & a_{mn}b_{11} & a_{mn}b_{1l} \\ a_{m1}b_{k1} & a_{m1}b_{kl} & a_{mn}b_{k1} & a_{mn}b_{kl} \end{bmatrix} \begin{bmatrix} c_{11}d_{11} & c_{11}d_{1s} & c_{1q}d_{11} & c_{1q}d_{1s} \\ c_{11}d_{r1} & c_{11}d_{rs} & c_{1q}d_{r1} & c_{1q}d_{rs} \\ c_{p1}d_{11} & c_{p1}d_{1s} & c_{pq}d_{11} & c_{pq}d_{1s} \\ c_{p1}d_{r1} & c_{p1}d_{rs} & c_{pq}d_{r1} & c_{pq}d_{rs} \end{bmatrix} =$$

$$= \left[\Sigma_{i=1}^{n}\Sigma_{j=1}^{r} a_{m'i}b_{k'j}c_{iq'}d_{js'}\right]$$

$$(AC) \otimes (BD) = \left(\begin{bmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn} \end{bmatrix} \begin{bmatrix} c_{11} & c_{1q} \\ c_{p1} & c_{pq} \end{bmatrix} \right) \otimes \left(\begin{bmatrix} b_{11} & b_{1l} \\ b_{k1} & b_{kl} \end{bmatrix} \begin{bmatrix} d_{11} & d_{1s} \\ d_{r1} & d_{rs} \end{bmatrix} \right) =$$

$$= \left[\Sigma_{i=1}^{n} a_{m'i}c_{iq'}\right] \otimes \left[\Sigma_{j=1}^{r} b_{k'j}d_{js'}\right] = \left[\Sigma_{i=1}^{n}\Sigma_{j=1}^{r} a_{m'i}b_{k'j}c_{iq'}d_{js'}\right] = (A \otimes B)(C \otimes D)$$

a2) $(A \otimes B)' = \begin{bmatrix} a_{11}B & a_{1n}B \\ a_{m1}B & a_{mn}B \end{bmatrix}' = \begin{bmatrix} a_{11}B' & a_{m1}B' \\ a_{1n}B' & a_{mn}B' \end{bmatrix} = (A' \otimes B')$

a3) $(A \otimes B)(A^{-1} \otimes B^{-1}) = (AA^{-1}) \otimes (BB^{-1}) = I \otimes I = I \quad \Rightarrow \quad (A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$

b) $y = (I \otimes X)\pi + u \qquad E(u|X) = 0 \qquad Var(u|X) = \Sigma \otimes I_N$

b1) $\hat{\pi}_{GLS} = \left((I \otimes X)'(\Sigma \otimes I_N)^{-1}(I \otimes X)\right)^{-1}\left((I \otimes X)'(\Sigma \otimes I_N)^{-1} y\right) =$

$= ((I \otimes X')(\Sigma^{-1} \otimes I_N)(I \otimes X))^{-1}((I \otimes X')(\Sigma^{-1} \otimes I_N) y) =$

$= ((\Sigma^{-1} \otimes X')(I \otimes X))^{-1}(\Sigma^{-1} \otimes X') y = (\Sigma^{-1} \otimes X'X)^{-1}(\Sigma^{-1} \otimes X') y =$

$\left(\Sigma\Sigma^{-1} \otimes (X'X)^{-1} X'\right) y = \left(I \otimes (X'X)^{-1} X'\right) y$

b2) $Var(\hat{\pi}_{GLS}|X) = \left(I \otimes (X'X)^{-1} X'\right)(\Sigma \otimes I_N)\left(I \otimes (X'X)^{-1} X'\right)' =$

$= \left(\Sigma \otimes (X'X)^{-1} X'\right)\left(I \otimes X (X'X)^{-1}\right) = \left(\Sigma \otimes (X'X)^{-1} X'X (X'X)^{-1}\right) = \Sigma \otimes (X'X)^{-1}$

b3) $y = (I \otimes X) G\theta + u \qquad E(u|X) = 0 \qquad Var(u|X) = \Sigma \otimes I_N \qquad A = \Sigma \otimes (X'X)^{-1}$

$\hat{\theta}_{GLS} = (G'(I \otimes X')(\Sigma^{-1} \otimes I_N)(I \otimes X) G)^{-1}(G'(I \otimes X')(\Sigma^{-1} \otimes I_N) y) =$

$= (G'(\Sigma^{-1} \otimes X'X) G)^{-1} G'(\Sigma^{-1} \otimes X') y = (G'A^{-1}G)^{-1} G'(\Sigma^{-1} \otimes X') y = (G'A^{-1}G)^{-1} G'A^{-1}\hat{\pi}_{GLS}$

Exercise 2

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \qquad Var(\varepsilon|X) = \begin{bmatrix} \sigma_{11}I_N & \sigma_{12}I_N \\ \sigma_{21}I_N & \sigma_{22}I_N \end{bmatrix}$$

$$\hat{\pi}_{GLS} = \left(\begin{bmatrix} 1 & x & 0 \\ 0 & 0 & 1 \end{bmatrix}' \begin{bmatrix} \sigma_{11}I_N & \sigma_{12}I_N \\ \sigma_{21}I_N & \sigma_{22}I_N \end{bmatrix}^{-1} \begin{bmatrix} 1 & x & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} *$$

$$\begin{bmatrix} 1 & x & 0 \\ 0 & 0 & 1 \end{bmatrix}' \begin{bmatrix} \sigma_{11}I_N & \sigma_{12}I_N \\ \sigma_{21}I_N & \sigma_{22}I_N \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \sigma^{11} & \overline{x}\sigma^{11} & \sigma^{12} \\ \overline{x}\sigma^{11} & \overline{x^2}\sigma^{11} & \overline{x}\sigma^{12} \\ \sigma^{21} & \overline{x}\sigma^{21} & \sigma^{22} \end{bmatrix}^{-1} \begin{bmatrix} \overline{y_1}\sigma^{11} + \overline{y_2}\sigma^{12} \\ \overline{xy_1}\sigma^{11} + \overline{xy_2}\sigma^{12} \\ \overline{y_1}\sigma^{21} + \overline{y_2}\sigma^{22} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}^{-1} = \frac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{bmatrix} = \begin{bmatrix} \sigma^{11} & \sigma^{12} \\ \sigma^{21} & \sigma^{22} \end{bmatrix}$$

$$\begin{bmatrix} \sigma^{11} & 0 & \sigma^{12} \\ 0 & \overline{x^2}\sigma^{11} & 0 \\ \sigma^{21} & 0 & \sigma^{22} \end{bmatrix} \begin{bmatrix} \widehat{\alpha_1} \\ \widehat{\beta} \\ \widehat{\alpha_2} \end{bmatrix} = \begin{bmatrix} \overline{y_1}\sigma^{11} + \overline{y_2}\sigma^{12} \\ \overline{xy_1}\sigma^{11} + \overline{xy_2}\sigma^{12} \\ \overline{y_1}\sigma^{21} + \overline{y_2}\sigma^{22} \end{bmatrix}$$

$\widehat{\alpha_1} = \overline{y_1}, \widehat{\alpha_2} = \overline{y_2}, \widehat{\beta} = \frac{\overline{xy_1}\sigma^{11} + \overline{xy_2}\sigma^{12}}{\overline{x^2}\sigma^{11}} \Rightarrow \widehat{\alpha_{iGLS}} = \widehat{\alpha_{iOLS}}, \widehat{\beta_{GLS}} = \widehat{\beta_{1OLS}} - \frac{\sigma_{12}}{\sigma_{22}}\widehat{\beta_{2OLS}}$

Exercise 3

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix}_i \Gamma = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}_i B + \begin{bmatrix} \varepsilon_1 & \varepsilon_2 \end{bmatrix}_i \qquad i = \overline{1, n}$$

$$Var(\varepsilon|X) = \begin{bmatrix} \sigma_{11}I_N & \sigma_{12}I_N \\ \sigma_{21}I_N & \sigma_{22}I_N \end{bmatrix} \qquad B = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix} \qquad \Gamma = \begin{bmatrix} 1 & -\gamma_2 \\ -\gamma_1 & 1 \end{bmatrix}$$

$$\Pi = B\Gamma^{-1} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix} \begin{bmatrix} 1 & -\gamma_2 \\ -\gamma_1 & 1 \end{bmatrix}^{-1} = \frac{1}{1-\gamma_1\gamma_2} \begin{bmatrix} \beta_{11} + \gamma_1\beta_{12} & \beta_{12} + \gamma_2\beta_{11} \\ \beta_{21} + \gamma_1\beta_{22} & \beta_{22} + \gamma_2\beta_{21} \\ \beta_{31} + \gamma_1\beta_{32} & \beta_{32} + \gamma_2\beta_{31} \end{bmatrix}$$

The system is not identified without restrictions since we only have 6 values for 8 parameters and none of the equations is separated. Need at least 2 restrictions.

1. $\Pi|_{\beta_{21}=0,\beta_{32}=0} = \frac{1}{1-\gamma_1\gamma_2} \begin{bmatrix} \beta_{11} + \gamma_1\beta_{12} & \beta_{12} + \gamma_2\beta_{11} \\ \gamma_1\beta_{22} & \beta_{22} \\ \beta_{31} & \gamma_2\beta_{31} \end{bmatrix}$ All are identified.
2. $\Pi|_{\beta_{12}=0,\beta_{22}=0} = \frac{1}{1-\gamma_1\gamma_2} \begin{bmatrix} \beta_{11} & \gamma_2\beta_{11} \\ \beta_{21} & \gamma_2\beta_{21} \\ \beta_{31} + \gamma_1\beta_{32} & \beta_{32} + \gamma_2\beta_{31} \end{bmatrix}$ Only γ_2 and β_{i2} are identified.
3. $\Pi|_{\gamma_1=0} = \begin{bmatrix} \beta_{11} & \beta_{12} + \gamma_2\beta_{11} \\ \beta_{21} & \beta_{22} + \gamma_2\beta_{21} \\ \beta_{31} & \beta_{32} + \gamma_2\beta_{31} \end{bmatrix}$ Only γ_1 and β_{i1} are identified.
4. $\Pi|_{\gamma_1=\gamma_2,\beta_{32}=0} = \frac{1}{1-\gamma^2} \begin{bmatrix} \beta_{11} + \gamma\beta_{12} & \beta_{12} + \gamma\beta_{11} \\ \beta_{21} + \gamma\beta_{22} & \beta_{22} + \gamma\beta_{21} \\ \beta_{31} & \gamma\beta_{31} \end{bmatrix}$ All are identified.
5. $\Pi|_{\sigma_{12}=0,\beta_{31}=0} = \frac{1}{1-\gamma_1\gamma_2} \begin{bmatrix} \beta_{11} + \gamma_1\beta_{12} & \beta_{12} + \gamma_2\beta_{11} \\ \beta_{21} + \gamma_1\beta_{22} & \beta_{22} + \gamma_2\beta_{21} \\ \gamma_1\beta_{32} & \beta_{32} \end{bmatrix}$ Only γ_1 and β_{i1} are identified.
6. $\Pi|_{\gamma_1=0,\sigma_{12}=0} = \begin{bmatrix} \beta_{11} & \beta_{12} + \gamma_2\beta_{11} \\ \beta_{21} & \beta_{22} + \gamma_2\beta_{21} \\ \beta_{31} & \beta_{32} + \gamma_2\beta_{31} \end{bmatrix}$ Only γ_1 and β_{i1} are identified.
7. $\Pi|_{\beta_{22}=1-\beta_{21}} = \frac{1}{1-\gamma_1\gamma_2} \begin{bmatrix} \beta_{11} + \gamma_1\beta_{12} & \beta_{12} + \gamma_2\beta_{11} \\ \beta_{21} + \gamma_1(1-\beta_{21}) & 1 - \beta_{21} + \gamma_2\beta_{21} \\ \beta_{31} + \gamma_1\beta_{32} & \beta_{32} + \gamma_2\beta_{31} \end{bmatrix}$ None are identified.
8. $\Pi|_{\beta_{21}=0,\beta_{22}=0,\beta_{31}=0,\beta_{32}=0} = \frac{1}{1-\gamma_1\gamma_2} \begin{bmatrix} \beta_{11} + \gamma_1\beta_{12} & \beta_{12} + \gamma_2\beta_{11} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ None are identified.
9. $\Pi|_{\beta_{21}=0,\beta_{22}=0,\beta_{31}=0,\beta_{32}=0,\beta_{11}=0} = \frac{1}{1-\gamma_1\gamma_2} \begin{bmatrix} \gamma_1\beta_{12} & \beta_{12} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ Only γ_1 is identified.

Exercise 4

$$A' = \begin{bmatrix} I-\Gamma \\ B \end{bmatrix}' = \begin{bmatrix} 0 & -\gamma_{21} & 0 & -\gamma_{41} & 0 & \beta_{21} & \beta_{31} & 0 & 0 \\ -\gamma_{12} & 0 & -\gamma_{32} & -\gamma_{42} & \beta_{12} & 1 & \beta_{32} & 0 & \beta_{52} \\ 0 & -\gamma_{23} & 0 & 0 & \beta_{13} & 0 & \beta_{33} & \beta_{43} & 0 \\ 0 & -\gamma_{24} & -\gamma_{34} & 0 & \beta_{14} & \beta_{24} & 0 & \beta_{44} & 0 \end{bmatrix}$$

$m = 4$ $R_1 = 5$ $R_2 = 3$ $R_3 = 5$ $R_4 = 4$ Equation 2 is not identified. Equations 1 and 3 are overidentified. Only $\gamma_{i1}, \gamma_{i3}, \gamma_{i4}, \beta_{j1}, \beta_{j3}, \beta_{j4}$ are identified.