Economics 201A, TA section 3 — Anton Cheremukhin

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1. **Envelope Theorem**

Let $F(a) = \max f(x,a)$ *s.t.* $g(x,a) \geq 0,\ x \geq 0,$ *Assume, the solution is differentiable.*

$\pounds(x,\lambda,a) = f(x,a) + \lambda g(x,a)$ *Then* $\frac{\partial F(a)}{\partial a_j} = \left.\frac{\partial \pounds}{\partial a_j}\right|_{x(a),\lambda(a)}$

Sketch of proof: $\frac{\partial F(a)}{\partial a_j} = \frac{\partial \pounds}{\partial x}\frac{\partial x}{\partial a_j} + \frac{\partial \pounds}{\partial \lambda}\frac{\partial \lambda}{\partial a_j} + \frac{\partial \pounds}{\partial a_j} = \frac{\partial \pounds}{\partial a_j} + 0 + 0 = \frac{\partial \pounds}{\partial a_j}$

$\frac{\partial \pounds}{\partial x_i} = 0$ or $x_i = 0$ and therefore $\frac{\partial x_i}{\partial a_j} = 0.$ Same for λ instead of x.

2. **Two problems.**

Primal problem	**Dual problem**
$\max_x \{u(x) \mid p \cdot x \leq I\}$	$\min_x \{p \cdot x \mid u(x) \geq U\}$
$u(x)$ - direct utility	$I = p \cdot x$ - income
$x(p,I)$ - marshallian demand	$x^c(p,U)$ - (hicksian) compensated demand
$v(p,I) = u(x(p,I))$ - indirect utility	$M(p,U) = p \cdot x^c(p,U)$ - expenditure
• continuous	• continuous
• homogeneous degree zero in (p,I)	• homogeneous of degree one in p
• strictly increasing in I	• strictly increasing in U
• decreasing in p	• increasing in p
• quasiconvex in (p,I)	• concave in p
Roy's identity: $x_i(p,I) = -\frac{\partial v(p,I)/\partial p_i}{\partial v(p,I)/\partial I}$	Sheppard's lemma: $\partial M(p,U)/\partial p_i = x_i^c(p,U)$

a) Roy's identity: Imagine $\pounds(x,\lambda,p,I) = u(x) + \lambda(I - px)$ $\quad v(p,I) = u(x(p,I))$

ENV: $\frac{\partial v(p,I)}{\partial I} = \frac{\partial \pounds}{\partial I} = \lambda$ $\quad \frac{\partial v(p,I)}{\partial p_i} = \frac{\partial \pounds}{\partial p_i} = -\lambda x_i$ Therefore, $x_i = -\frac{\partial v(p,I)}{\partial I} / \frac{\partial v(p,I)}{\partial p_i}$.

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b) Sheppard's lemma: $\pounds(x,\lambda,p,U) = px + \lambda(u(x) - U)$ $\quad M(p,U) = p \cdot x^c(p,U)$

ENV: $\frac{\partial M(p,U)}{\partial p_i} = \frac{\partial \pounds}{\partial p_i} = x_i^c(p,U)$

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c) $M(p,U)$ concave: $M(tp^1 + (1-t)p^2, U) = \min_x \{(tp^1 + (1-t)p^2) \cdot x \mid u(x) \geq U\} \geq$

$\geq \min_x \{(1-t)p^2 \cdot x \mid u(x) \geq U\} + \min_x \{tp^1 \cdot x \mid u(x) \geq U\} = tM(p^1, U) + (1-t)M(p^2, U)$

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3. **Equivalence Relationships**:

$M(p, v(p,I)) = I$ $\qquad v(p, M(p,U)) = U$

$x_i(p,I) = x_i^c(p, v(p,I))$ $\qquad x_i^c(p,U) = x_i(p, M(p,U))$

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Differentiating the last and using Sheppard's lemma:

$\frac{\partial x_i^c(p,U)}{\partial p_j} = \frac{\partial x_i(p,I)}{\partial p_j} + \frac{\partial M(p,U)}{\partial p_j}\frac{\partial x_i(p,I)}{\partial I} = \frac{\partial x_i(p,I)}{\partial p_j} + x_j \frac{\partial x_i(p,I)}{\partial I}$. Therefore:

Slutsky equation: $\frac{\partial x_i(p,I)}{\partial p_j} = \left.\frac{\partial x_i^c(p,U)}{\partial p_j}\right|_U - x_j(p,I)\frac{\partial x_i(p,I)}{\partial I} = SE + IE$

4. **Example: Midterm 1998, Question 1**

Let $u(x) = x_1^{\alpha} x_2^{\beta}$, $p = (p_1, p_2)$, $w = (w_1, w_2)$.
a) Find demands
Cobb-Douglas: $p_i x_i = \frac{\alpha}{\alpha+\beta}(p_1 w_1 + p_2 w_2)$
Therefore, $x_1(p, I) = \frac{\alpha}{\alpha+\beta}\frac{p_1 w_1 + p_2 w_2}{p_1}$ $\qquad x_2(p, I) = \frac{\beta}{\alpha+\beta}\frac{p_1 w_1 + p_2 w_2}{p_2}$
b) Derive the expenditure function
$v(p, I) = \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{p_1 w_1 + p_2 w_2}{\alpha+\beta}\right)^{\alpha+\beta} = U$
Therefore, $M(p, U) = (\alpha+\beta)\left(\frac{p_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{p_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} U^{\frac{1}{\alpha+\beta}}$
$x_1^c(p_1, p_2, U) = \frac{1}{\alpha+\beta}\frac{\alpha}{p_1}(\alpha+\beta)\left(\frac{p_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{p_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} U^{\frac{1}{\alpha+\beta}} = \left(\frac{\alpha}{p_1}\frac{p_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} U^{\frac{1}{\alpha+\beta}}$
$x_2^c(I, p_1, p_2) = \frac{\beta}{\alpha+\beta}\frac{p_1 w_1 + p_2 w_2}{p_2} = \left(\frac{p_1}{\alpha}\frac{\beta}{p_2}\right)^{\frac{\alpha}{\alpha+\beta}} U^{\frac{1}{\alpha+\beta}}$
c) Prove that $x_1 = w_1 - p_1 \frac{\partial v(p,I)/\partial p_1}{\partial v(p,I)/\partial w_1}$
$\pounds(p, w) = u(x) + \lambda(p_1 w_1 + p_2 w_2 - p_1 x_1 - p_2 x_2)$
Using Roy's identity: $-\frac{\partial v(p,I)/\partial p_1}{\partial v(p,I)/\partial w_1} = -\frac{\partial \pounds(p,I)/\partial p_1}{\partial \pounds(p,I)/\partial w_1} = -\frac{\lambda(w_1 - x_1)}{\lambda p_1} = \frac{x_1 - w_1}{p_1}$
Therefore, $x_1 = w_1 - p_1 \frac{\partial v(p,I)/\partial p_1}{\partial v(p,I)/\partial w_1}$.
d) When is the indirect utility convex in endowments?
$v(p, k \cdot w) = \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{p_1 k w_1 + p_2 k w_2}{\alpha+\beta}\right)^{\alpha+\beta} = v(p, w) k^{\alpha+\beta}$
$\frac{\partial^2}{\partial k^2} k^{\alpha+\beta} = k^{\alpha+\beta-2}(\alpha+\beta)(\alpha+\beta-1)$
So, if $\alpha+\beta \geq 1$, $v(p, w)$ is convex in w.

5. **Elasticities**:

$E[x, I] = \frac{I}{x}\frac{\partial x}{\partial I}$ $\qquad$ income share: $k_i = \frac{p_i x_i}{I}$
Engel aggregation (Lemma, 4-2 page 4): $\sum_{i=1}^{n} k_i E[x_i, I] = 1$
"weighted income elasticities of demand sum up to 1"
Proof: Use $I = px(p, I)$
$1 = \frac{\partial I}{\partial I} = \frac{\partial(px(p,I))}{\partial I} = \sum_{i=1}^{n} \frac{p_i \partial x_i}{\partial I} = \sum_{i=1}^{n} \frac{p_i x_i}{I}\frac{I}{x_i}\frac{\partial x_i}{\partial I} = \sum_{i=1}^{n} k_i E[x_i, I]$
$E[x_i, p_j]$ - cross-price elasticities, $\qquad E[x_i, p_i]$ - own-price elasticity
Slutsky equation: $E[x_i, p_i] = E[x_i^c, p_i] - k_i E[x_i, I]$
Proof: $\frac{\partial x_i}{\partial p_j} = \frac{\partial x_i^c}{\partial p_j} - x_j \frac{\partial x_i}{\partial I}$ $\qquad \frac{p_j}{x_i}\frac{\partial x_i}{\partial p_j} = \frac{p_j}{x_i}\frac{\partial x_i^c}{\partial p_j} - \frac{I}{I}\frac{p_j}{x_i} x_j \frac{\partial x_i}{\partial I}$
$\frac{p_j}{x_i}\frac{\partial x_i}{\partial p_j} = \frac{p_j}{x_i}\frac{\partial x_i^c}{\partial p_j} - \frac{p_j x_j}{I}\frac{I}{x_i}\frac{\partial x_i}{\partial I}$ $\qquad E[x_i, p_j] = E[x_i^c, p_j] - k_j E[x_i, I]$
Cournot aggregation (Lemma, 4-2 page 10): $\sum_{i=1}^{n} k_i E[x_i, p_j] + k_j = 0$ for all j.
For the compensated demand if follows that $\sum_{i=1}^{n} k_i E[x_i^c, p_j] = 0$
"weighted price elasticities of compensated demand sum up to zero"
Proof: Use $I = px(p, I)$
$0 = \frac{\partial I}{\partial p_j} = \frac{\partial(px(p,I))}{\partial p_j} = x_j + \sum_{i=1}^{n} p_i \frac{\partial x_i(p,I)}{\partial p_j}$
$0 = \frac{p_j x_j}{I} + \sum_{i=1}^{n} p_i \frac{\partial x_i(p,I)}{\partial p_j}\frac{x_i}{x_i}\frac{p_j}{I} = k_j + \sum_{i=1}^{n} \frac{p_i x_i}{I}\frac{\partial x_i(p,I)}{\partial p_j}\frac{p_j}{x_i}$
Use Slutsky and Engel aggregation:

$0=\sum_{i=1}^{n} k_i E\left[x_i,p_j\right]+k_j=k_j+\sum_{i=1}^{n} k_i\left(E\left[x_i^c,p_j\right]-k_j E\left[x_i,I\right]\right)=$
$=\sum_{i=1}^{n} k_i\left(E\left[x_i^c,p_j\right]\right)+k_j-\sum_{i=1}^{n} k_i k_j E\left[x_i,I\right]=\sum_{i=1}^{n} k_i\left(E\left[x_i^c,p_j\right]\right)$
Elasticity of substitution (2 goods): $E\left[\frac{x_2^c}{x_1^c},p_1\right]=p_1\frac{x_1^c}{x_2^c}\frac{\partial}{\partial p_1}\left(\frac{x_2^c}{x_1^c}\right)$
$E\left[\frac{x_2^c}{x_1^c},p_1\right]=p_1\frac{\partial}{\partial p_1}\ln\frac{x_2^c}{x_1^c}=p_1\frac{\partial}{\partial p_1}\left(\ln x_2^c-\ln x_1^c\right)=E\left[x_2^c,p_1\right]-E\left[x_1^c,p_1\right]$
Cournot: $0=k_1E\left[x_1^c,p_1\right]+k_2E\left[x_2^c,p_1\right] \quad \Rightarrow$
$0=k_1E\left[x_1^c,p_1\right]+k_2\left(E\left[\frac{x_2^c}{x_1^c},p_1\right]+E\left[x_1^c,p_1\right]\right)$
$0=k_1\left(-E\left[\frac{x_2^c}{x_1^c},p_1\right]+E\left[x_2^c,p_1\right]\right)+k_2E\left[x_2^c,p_1\right]$
Therefore, $E\left[\frac{x_2^c}{x_1^c},p_1\right]=-\frac{E\left[x_1^c,p_1\right]}{k_2}=\frac{E\left[x_2^c,p_1\right]}{k_1}=-E\left[\frac{x_1^c}{x_2^c},p_1\right]=\sigma$
Using this, one can show, that $E\left[x_1^c,p_1\right]=-k_2\sigma \qquad E\left[x_2^c,p_1\right]=k_1\sigma$.
Using Slutsky, $E\left[x_1,p_1\right]=-\left(1-k_1\right)\sigma-k_1E\left[x_i,I\right]$ - decomposition of own price elasticity.

6. **Example: CES.**

CES: $U(c)=\left(\sum_{i=1}^{n} c_i^{1-\frac{1}{\sigma}}\right)^{\frac{1}{1-\frac{1}{\sigma}}}$, $\sigma>0$, $\sigma\neq 1$

$$\frac{\partial}{\partial c_i}U(c)=\frac{1}{1-\frac{1}{\sigma}}\left(\sum_{i=1}^{n} c_i^{1-\frac{1}{\sigma}}\right)^{\frac{1}{1-\frac{1}{\sigma}}-1}\left(1-\frac{1}{\sigma}\right)c_i^{-\frac{1}{\sigma}}=\left(\sum_{i=1}^{n} c_i^{1-\frac{1}{\sigma}}\right)^{\frac{\frac{1}{\sigma}}{1-\frac{1}{\sigma}}}c_i^{-\frac{1}{\sigma}}=\left(\frac{U(c)}{c_i}\right)^{\frac{1}{\sigma}}$$

FOC: $\frac{\partial}{\partial c_i}U(c)=\lambda p_i \quad \Rightarrow \quad \frac{\frac{\partial}{\partial c_i}U(c)}{p_i}=\frac{\frac{\partial}{\partial c_j}U(c)}{p_j} \quad \Leftrightarrow$

$$\frac{\left(\frac{U(c)}{c_i}\right)^{\frac{1}{\sigma}}}{p_i}=\frac{\left(\frac{U(c)}{c_j}\right)^{\frac{1}{\sigma}}}{p_j} \quad \Leftrightarrow \quad p_i c_i^{\frac{1}{\sigma}}=p_j c_j^{\frac{1}{\sigma}} \quad \Leftrightarrow \quad \left(\frac{p_i}{p_j}\right)^{\sigma}=\frac{c_j}{c_i}$$

$$\sum_{j=1}^{n} p_j c_j=\sum_{j=1}^{n} p_j c_i\left(\frac{p_i}{p_j}\right)^{\sigma}=c_i\sum_{j=1}^{n}\left(p_j\right)^{1-\sigma}\left(p_i\right)^{\sigma}=I \quad \Rightarrow \quad c_i(p,I)=\boxed{\frac{I}{\left(p_i\right)^{\sigma}\sum_{j=1}^{n}\left(p_j\right)^{1-\sigma}}}$$

$$U(c(p,I))=\left(\sum_{i=1}^{n} c_i^{1-\frac{1}{\sigma}}(p,I)\right)^{\frac{1}{1-\frac{1}{\sigma}}}=I\left(\sum_{i=1}^{n}\left(\left(p_i\right)^{\sigma}\sum_{j=1}^{n}\left(p_j\right)^{1-\sigma}\right)^{\frac{1}{\sigma}-1}\right)^{\frac{1}{1-\frac{1}{\sigma}}}$$

$$=I\left(\sum_{j=1}^{n}\left(p_j\right)^{1-\sigma}\right)^{-1}\left(\sum_{i=1}^{n}\left(p_i\right)^{1-\sigma}\right)^{\frac{1}{1-\frac{1}{\sigma}}} \qquad v(p,I)=\boxed{I\left(\sum_{i=1}^{n}\left(p_i\right)^{1-\sigma}\right)^{\frac{1}{\sigma-1}}}$$

$$M(p,U)=\boxed{U\Big/\left(\sum_{i=1}^{n}\left(p_i\right)^{1-\sigma}\right)^{\frac{1}{\sigma-1}}} \qquad c_i^c(p,I)=\boxed{\frac{U}{\left(p_i\right)^{\sigma}\left(\sum_{i=1}^{n}\left(p_i\right)^{1-\sigma}\right)^{\frac{\sigma}{\sigma-1}}}} \qquad \frac{c_j}{c_i}=\frac{c_j^c}{c_i^c}=\left(\frac{p_i}{p_j}\right)^{\sigma}$$

$E\left[\frac{c_j^c}{c_i^c},p_i\right]=p_i\frac{\partial}{\partial p_i}\ln\frac{c_j^c}{c_i^c}=p_i\frac{\partial}{\partial p_i}\ln\left(\frac{p_i}{p_j}\right)^{\sigma}=p_i\frac{\partial}{\partial p_i}\sigma\left(\ln p_i-\ln p_j\right)=\sigma$
CES utility - constant elasticity of substitution.