TREE-CUTTING PROBLEM Anton Cheremukhin

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The tree-cutting problem (N. L. Stokey, R. E. Lucas (1989), Problem 5.5)

Consider a tree whose growth is described by the function h. That is, if k_t is the size of the tree in period t, then $k_{t+1} = h(k_t), t = 0, 1, \ldots$. Assume that the price of wood p, and the interest rate r, are both constant over time; let $p = 1$ and $\beta = 1/(1 + r)$. Assume that it is costless to cut down the tree.

a) If the tree cannot be replanted, present value maximization leads to the functional equation $v(k) = \max\{k, \beta v(h(k))\}$. Under what assumptions about h is there a simple rule describing when the tree ought to be cut down and sold?

(b) Suppose that when the tree is cut down, another can be planted in its place and so on. Assume that the cost of replanting, $c \geq 0$, is constant over time. Under what assumptions about h and c is there a simple rule describing when the trees should be harvested?

Solution

(a) $v(k) = \max\{k, \beta v(h(k))\}$.
Assumptions:
A1: Set $\mathbf{h}(0) \geq \mathbf{0}$ because the tree is growing.
A2: Let the **state space be bounded**: $[0, \overline{k}]$ where $\overline{k}$ is big.
A3: Let $\mathbf{h}(\mathbf{k})$ **be increasing**,
A4: $\mathbf{h}(\mathbf{k})$ **crosses** $\mathbf{k}/\beta$ **only once**: **from above (except possibly zero)**.
i.e. $\exists k_0$: $\forall k > k_0 : h(k) < k/\beta$. $\forall$ and $\forall k < k_0 : h(k) > k/\beta$.
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Blackwell conditions hold:
B1) If $f(x) \leq g(x)$ for all x, then $\beta f(h(k)) \leq \beta g(h(k))$ and $\max\{k, \beta f(h(k))\} \leq \max\{k, \beta g(h(k)\}$
B2) $\max\{k, \beta[f(h(k)) + a]\} \leq \max\{k + \beta a, \beta[f(h(k)) + a]\} = \max\{k, [f(h(k))]\} + \beta a$
Hence, there exists a unique value function $v(.)$ (fixed point).
$v(.)$ is increasing: If $f(k') \leq f(k'')$ for all $k' \leq k''$, then $\beta f(h(k')) \leq \beta f(h(k''))$ and $\max\{k, \beta f(h(k'))\} \leq \max\{k, \beta f(h(k'')\}$. Hence, limiting $v(.)$ is also increasing.
Proof of existence of a simple rule:

(1) Assume $\forall k > k_0 : f(k) \leq k$. Then $\max\{k, \beta[f(h(k))]\} \leq \max\{k, \beta f(k/\beta)\} = \beta \max\{k/\beta, f(k/\beta)\} = \beta\{k/\beta\} = k$, which means that once we start with some function f satisfying the assumption all the future iterations also stick to it. So does the fixed point. Hence, $\forall k > k_0 : v(k) \leq k$. Since $v(k) \geq k$ by definition, we can state that $\forall k > k_0 : v(k) = k$. Hence, you should cut down the tree for all $k > k_0$.

(2) Assume $\forall k < k_0 : f(k) > k$. Then $\max\{k, \beta[f(h(k))]\} \geq \max\{k, \beta f(k/\beta)\} = \beta \max\{k/\beta, f(k/\beta)\} = \beta f(k/\beta) > \beta k/\beta = k$, which means that once we start with some function f satisfying the assumption all the future iterations also stick to it. So does the fixed point. Hence, $\forall k < k_0 : v(k) > k$. Hence, you should not cut down the tree for all $k < k_0$.

So, we get a simple **decision rule**: cut the tree if $k \geq k_0 : h(k_0) = k_0/\beta$.

Notice that we didn't mention continuity or concavity. If the function is concave there cannot be more than one crossing. However, there could be no crossing at all in which case we should either never cut the tree or cut it anyway.

(b) $v(k) = \max\{k - c + \beta v(h(0)), \beta v(h(k)), k\}$
Assumptions:
A1: Set $\mathbf{h}(0) \geq \mathbf{c}/\beta > \mathbf{0}$ because the tree is growing from zero-size.
A2: Let the **state space be bounded**: $[0, \overline{k}]$ where $\overline{k}$ is big.
A3: Let $\mathbf{h}(\mathbf{k})$ **be increasing**.
A4: $\mathbf{h}(\mathbf{k}) - \mathbf{h}(\mathbf{0})$ **crosses** $(\mathbf{k} - \mathbf{c})/\beta$ **only once**: **from above (except possibly zero)**.
i.e. $\exists k_0$: $\forall k > k_0 : h(k) < h(0) + (k - c)/\beta$. $\forall$ and $\forall k < k_0 : h(k) > h(0) + (k - c)/\beta$.
.
The problem satisfies Blackwell conditions and there exists a unique fixed point:
B1) If $f(x) \leq g(x)$ for all x, then $\beta f(h(k)) \leq \beta g(h(k))$ and
$\max\{k - c + \beta f(h(0)), \beta f(h(k)), k\} \leq \max\{k - c + \beta g(h(0)), \beta g(h(k), k\}$
B2) $\max\{k - c + \beta[f(h(0) + a]), \beta[f(h(k)) + a], k\} =$
$\beta a + \max\{k - c + \beta f(h(0)), \beta f(h(k)), k - \beta a\} \leq \max\{k - c + \beta f(h(0)), \beta f(h(k)), k\} + \beta a$
$v(.)$ is increasing: If $f(k') \leq f(k'')$ for all $k' \leq k''$, then $\beta f(h(k')) \leq \beta f(h(k''))$ and
$\max\{k' - c + \beta f(h(0)), \beta f(h(k')), k'\} \leq \max\{k'' - c + \beta f(h(0)), \beta f(h(k'')), k''\}$.
Given that the initial guess is increasing, the limiting $v(.)$ is also increasing.
Using A1 we show that $v(h(0)) \geq h(0) \geq c/\beta$:
Take some f such that $f(h(0)) \geq h(0) \geq c/\beta$. Then
$\max\{h(0) - c + \beta f(h(0)), \beta f(h(h(0))), h(0)\} \geq \max\{h(0) - c + \beta f(h(0)), \beta f(h(0)), h(0)\}$
$\geq \max\{h(0) - c + \beta h(0), \beta h(0), h(0)\} \geq h(0) \geq c/\beta$
Hence, by induction $v(h(0)) \geq h(0) \geq c/\beta$ and $[-c + \beta v(h(0))]$ is a non-negative constant.
Therefore, it is always better to replant the tree and the third option is dominated:

$$\boxed{v(k) = \max\{k - c + \beta v(h(0)), \beta v(h(k))\}}$$

Proof of existence of a simple rule:
(1) Suppose $\forall k > k_0 : f(k) \leq \beta f(h(0)) - c + k$. Then
$Tf(k) = \max\{k - c + \beta f(h(0)), \beta f(h(k))\} \leq \max\{k - c + \beta f(h(0)), \beta(\beta f(h(0)) - c + h(k))\}$
$\leq \max\{k - c + \beta f(h(0)), \beta(\beta f(h(0)) - c) + \beta h(0) - c + k)\} \leq$
$k - c + \beta f(h(0)) + (1 - \beta)\beta \max\{0, h(0) - f(h(0))\} = k - c + \beta f(h(0))$, which means that once we start with some function f satisfying the assumption all the future iterations also stick to it. So does the fixed point. Hence, $\forall k > k_0 : v(k) \leq \beta v(h(0)) - c + k$. Since $v(k) \geq \beta v(h(0)) - c + k$ by definition, we can state that $\forall k > k_0 : v(k) = \beta v(h(0)) - c + k$. Hence, you should cut down and replant for all $k > k_0$.

(2) Suppose now that $\forall k < k_0 : f(k) > \beta f(h(0)) - c + k$. Then
$\max\{k - c + \beta f(h(0)), \beta f(h(k))\} \geq \max\{k - c + \beta f(h(0)), \beta(\beta f(h(0)) - c + h(k))\} >$
$\max\{k - c + \beta f(h(0)), \beta(\beta f(h(0)) - c + h(0) + (k - c)/\beta)\} =$
$k - c + \beta f(h(0)) + \beta \max\{0, h(0) - c\} = k - c + \beta f(h(0))$, which means that once we start with some function f satisfying the assumption all the future iterations also stick to it. So does the fixed point. Hence, $\forall k < k_0 : v(k) > k - c + \beta f(h(0))$. Hence, you should not cut down the tree for all $k < k_0$.

So, we get a simple **decision rule**: cut the tree if $k \geq k_0 : h(k_0) = h(0) + (k_0 - c)/\beta$.

Again we didn't mention continuity or concavity. If the function is concave there cannot be more than one crossing. However, there could be no crossing at all in which case we should either never cut the tree or cut it every period.