Economics 201A, Homework 4, Answer Key. Anton Cheremukhin

December 7, 2006

1. **Certainty Equivalent**

$E\widetilde{x}=0 \qquad Var\widetilde{x}=\sigma^2 \qquad$ Certainty equivalence: $Ev\left(w+\widetilde{x}\right)=v\left(w-\alpha\right)$

$Ev\left(w+\widetilde{x}\right)\approx v\left(w\right)+v'\left(w\right)E\widetilde{x}+\frac{1}{2}v''\left(w\right)E\left(\widetilde{x}\right)^2=v\left(w\right)+v''\left(w\right)\frac{\sigma^2}{2}\approx v\left(w-\alpha\right)$

$\Rightarrow \qquad v''\left(w\right)\frac{\sigma^2}{2}\approx-\frac{v(w)-v(w-\alpha)}{\alpha}\alpha\approx-\alpha v'\left(w\right)$ Therefore, $\alpha\approx-\frac{v''(w)}{v'(w)}\frac{\sigma^2}{2}=A\left(w\right)\frac{\sigma^2}{2}$ - proportional to the degree of absolute risk aversion and to the amount of risk.

2. **Equilibrium with CES preferences and identical beliefs**

a-b) $\frac{p_s}{p_t}=MRS_{s,t}=\left.\frac{\pi_s v'(c_s)}{\pi_t v'(c_t)}\right|_{(w_s,w_t)}=\frac{\pi_s v'(w_s)}{\pi_t v'(w_t)}>\frac{\pi_s}{\pi_t}$ if $w_s<w_t$ because $v'\left(w\right)$ is a decreasing function. Agents have identical beliefs and identical homothetic utilities over contingent goods, so the RA argument holds.

c) Agents can achieve the same outcome under incomplete markets, if the total endowment is a linear combination of the assets' payoffs.

d) The fact that agents choose proportional consumptions depends only on their preference structure (identical, homothetic). So, if the total consumption in equilibrium is a linear combination of asset payoffs, then the result will hold. $\Sigma c=\Sigma w+\Sigma y$. So, we need this sum to be represented by assets.

3. **State Claims and Asset Markets Equilibrium**

There are 30 agents, each has a plantation. 10 plantations are stable and produce 10 units. 20 plantations produce 15 with probability $\frac{1}{3}$ and 5 units with probability $\frac{2}{3}$. $v\left(c\right)=-1/c$

a) So again, we have 30 identical homothetic agents, can use the representative agent framework. The total endowment is $\left(10*10+20*15,10*10+20*5\right)=\left(400,200\right).$ $v'\left(c\right)=1/c^2$

$\frac{p_1}{p_2}=MRS_{1,2}=\left.\frac{\pi_1 v'(c_1)}{\pi_2 v'(c_2)}\right|_{(w_1,w_2)}=\frac{1*\frac{1}{400^2}}{2*\frac{1}{200^2}}=\frac{1}{8}$ i.e. $p=\left(1,8\right)$

b) If we treat each plantation as an asset, then there are two assets A and B with payoffs $z_A=\left(10,10\right)$ and $z_B=\left(15,5\right).$ $\qquad p_A=1*10+8*10=90 \qquad p_B=1*15+8*5=55$

c) The payoff of investing in q_A units of asset A and q_B units of asset B is $c_1=10q_A+15q_B$ in state 1 and $c_2=10q_A+5q_B$ in state 2. So the utility of the representative agent will be $U=\Sigma\pi_s v\left(c_s\right)=\pi_1 v\left(10q_A+15q_B\right)+\pi_2 v\left(10q_A+5q_B\right)$

d) She will maximize this utility subject to her budget constraint in terms of assets: $p_Aq_A+p_Bq_B=p_A\overline{q}_A+p_B\overline{q}_B$, where $\overline{q}_A$ and $\overline{q}_B$ are her endowments of assets. The first order conditions will be $10\pi_1 v'\left(c_1\right)+10\pi_2 v'\left(c_2\right)=\lambda p_A \qquad 15\pi_1 v'\left(c_1\right)+5\pi_2 v'\left(c_2\right)=\lambda p_B$

Dividing one by the other we get the desired result: $\frac{p_A}{p_B}=\frac{10\pi_1+10\pi_2\frac{v'(10q_A+5q_B)}{v'(10q_A+15q_B)}}{15\pi_1+5\pi_2\frac{v'(10q_A+5q_B)}{v'(10q_A+15q_B)}}$

e) The market clearing implies $q=\overline{q}=\left(10,20\right)$. Substituting it into the above expression we get: $\frac{p_A}{p_B}=\frac{10\pi_1+10\pi_2\frac{v'(10q_A+5q_B)}{v'(10q_A+15q_B)}}{15\pi_1+5\pi_2\frac{v'(10q_A+5q_B)}{v'(10q_A+15q_B)}}=\frac{10\frac{(10*10+5*20)^2}{(10*10+15*20)^2}+10*2}{15\frac{(10*10+5*20)^2}{(10*10+15*20)^2}+5*2}=\frac{90}{55}$

f) The price is the same as when we trade in contingent claims. This the result of the complete asset markets: there are as many assets as states (two of each), and the asset payoffs are linearly independent.

4. **State Claims Market Equilibrium**

a) PE: $MRS_A = MRS_B$ and $c_s^A + c_s^B = w_s$ (resource constraints).
$\frac{2+c_2^A}{2+c_1^A} = \frac{2+c_2^B}{2+c_1^B} = \frac{4+w_2}{4+w_1} = \frac{40}{20} = 2 = \frac{p_1}{p_2}, \qquad c_2^h = 2 + 2c_1^h$
b) Budget constraint of A: $2c_1^A + c_2^A = 2*4+9 = 17$, also $c_2^A = 2+2c_1^A$ from PE.
So, $2c_1^A + 2 + 2c_1^A = 17$. Hence, $c^A = \left(\frac{15}{4}, \frac{19}{2}\right)$.
c) The plantations give proportional payoffs: $(12, 27) = 3*(4,9)$. So asset markets are incomplete. Also, agent's preferences are not homothetic, so the choices are not on the diagonal. Asset markets assess only choices on the diagonal, while the optimal consumption point is not on it. That's why trading shares in the two plantations does not help in the reallocation of risk.
d) The payoffs of the two stocks are $z_P = (3,3)$ and $z_O = (1,6)$. Under complete markets their market values would be: $p_P = 3*2+3*1 = 9 \qquad p_O = 1*2+6*1 = 8$
e) The payoffs of these two assets are now linearly independent, so one can assess any point in the Edgeworth box. We still have asset B by the way. So there are three independent assets and two states. Therefore, trading in an asset market is equivalent to trading in contingent claims and yields the desired outcome (risk reallocation).

5. **Time and Uncertainty**

a) There are six contingent goods in the economy, and the total endowment is $(81, 36, 100, 25, 49, 121)$. The agent's utility over contingent goods can be written as: $U(c) = \sqrt{c_1} + 2\sqrt{c_2} + \frac{2}{3}\left(\sqrt{c_3} + 2\sqrt{c_4}\right) + \frac{1}{3}\left(\sqrt{c_5} + 2\sqrt{c_6}\right)$. It is homothetic, so we can use the representative agent.
$\frac{p_2}{p_1} = MRS_{2,1} = \left.\frac{2\sqrt{c_1}}{\sqrt{c_2}}\right|_{(81,36)} = \frac{2*9}{6} = 3 \qquad \frac{p_3}{p_1} = MRS_{2,1} = \left.\frac{\frac{2}{3}\sqrt{c_1}}{\sqrt{c_3}}\right|_{(81,100)} = \frac{\frac{2}{3}*9}{10} = \frac{3}{5}$
$\frac{p_4}{p_1} = MRS_{2,1} = \left.\frac{\frac{4}{3}\sqrt{c_1}}{\sqrt{c_4}}\right|_{(81,25)} = \frac{\frac{4}{3}*9}{5} = \frac{12}{5} \qquad \frac{p_5}{p_1} = MRS_{2,1} = \left.\frac{\frac{1}{3}\sqrt{c_1}}{\sqrt{c_5}}\right|_{(81,49)} = \frac{\frac{1}{3}*9}{7} = \frac{3}{7}$
$\frac{p_6}{p_1} = MRS_{2,1} = \left.\frac{\frac{2}{3}\sqrt{c_1}}{\sqrt{c_6}}\right|_{(81,121)} = \frac{\frac{2}{3}*9}{11} = \frac{6}{11} \qquad \Rightarrow \qquad p = \left(\frac{1}{3}, 1, \frac{1}{5}, \frac{4}{5}, \frac{1}{7}, \frac{2}{11}\right)$
b) The technology allow us to transfer commodity 2 into both commodities 4 and 6. The production set of the firm is equivalent to: $\{y = (0, -q, 0, q, 0, q), q \geq 0\}$.
Profit of the firm is: $\pi(p) = py = (p_4 + p_6 - p_2)\, q$.
c) Under the prevailing prices $p_4 + p_6 - p_2 = \frac{4}{5} + \frac{2}{11} - 1 = -\frac{1}{55} < 0$. Production is not profitable. There is no incentive to produce, hence, in WE the technology won't be used, and it remains the same.
d) If a technology allowed us to transfer commodity 1 into both commodities 3 and 5, there would be incentives to produce: $\pi(p) = py = (p_3 + p_5 - p_1)\, q = \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{3}\right) q = \frac{q}{105} > 0$. Therefore, the technology would alter the equilibrium. To find the new consumption vector we maximize total utility over q subject to: $c_1 = w_1 - q$, $c_3 = w_3 + q$, $c_5 = w_5 + q$, $q > 0$.
$U(c) = \sqrt{w_1 - q} + 2\sqrt{w_2} + \frac{2}{3}\left(\sqrt{w_3 + q} + 2\sqrt{w_4}\right) + \frac{1}{3}\left(\sqrt{w_5 + q} + 2\sqrt{w_6}\right) \to \max_q$
This is equivalent to: $\sqrt{81 - q} + \frac{2}{3}\sqrt{100 + q} + \frac{1}{3}\sqrt{49 + q} \to \max_q$
FOC: $\frac{3}{\sqrt{81-q}} = \frac{2}{\sqrt{100+q}} + \frac{1}{\sqrt{49+q}}$, Solution is: $q = 2.1186$.
e) If there is storage, than in equilibrium prices must be such that the firm has zero profit (it has constant returns to scale). Therefore, it must be the case, that $p_3 + p_5 = p_1$.
f) If a technology is introduced only for good 2, then $\frac{p_4}{p_3} = MRS_{4,3} = \left.\frac{2\sqrt{c_3}}{\sqrt{c_4}}\right|_{c_3=100, c_4=25} = 4$
$\frac{p_6}{p_5} = MRS_{6,5} = \left.\frac{2\sqrt{c_5}}{\sqrt{c_6}}\right|_{c_5=49, c_6=121} = \frac{14}{11}$. So the relative price of commodity 1 would be $\frac{1}{4}$ in state 1 and $\frac{11}{14}$ in state 2.

If a technology is introduced for both goods, then $\frac{p_4}{p_3} = MRS_{4,3} = \left.\frac{2\sqrt{c_3}}{\sqrt{c_4}}\right|_{c_3=100+2.1186, c_4=25} = 4.0421$ $\quad \frac{p_6}{p_5} = MRS_{6,5} = \left.\frac{2\sqrt{c_5}}{\sqrt{c_6}}\right|_{c_5=49+2.1186, c_6=121} = 1.3000$. So the relative price of commodity 1 would be $\frac{1}{4.04}$ in state 1 and $\frac{1}{1.3}$ in state 2.

g) In the first setting there are 3 spot markets and 2 independent assets, which altogether create 5 relative prices. The dimension of the state claims market is 6, so there are 5 relative prices. The dimensions are the same, and independence of all the assets in the first setting implies that the first setting spans the whole set of outcomes. So it is equivalent to complete markets.