Economics 201A, Homework 3, Answer Key. Anton Cheremukhin

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1. Walrasian Equilibrium Prices

$U^A = \left(x_1^A - \alpha_1^A\right)^2 \left(x_2^A - \alpha_2^A\right) \sim 2\ln\left(x_1^A - \alpha_1^A\right) + \ln\left(x_2^A - \alpha_2^A\right) \qquad \alpha^A = (2,2)$

$U^B = \left(x_1^B - \alpha_1^B\right) \left(x_2^B - \alpha_2^B\right)^2 \sim \ln\left(x_1^B - \alpha_1^B\right) + 2\ln\left(x_2^B - \alpha_2^B\right) \qquad \alpha^B = (3,1) \qquad w_x = (13,10)$

a) Redefine variables: $y_i^J = x_i^J - \alpha_i^J$.

Then, $U^A \sim 2\ln y_1^A + \ln y_2^A \qquad U^B \sim \ln y_1^B + 2\ln y_2^B \qquad w_y = (8,7)$

In the small Edgeworth box, corresponding to the new coordinates, both preferences are homothetic, but preferences of A are more intensive in the first commodity while preferences of B are more intensive in the second commodity. That implies that the PE frontiere will go below the diagonal through both corners of the box. Also note, that the corresponding MRS's are continous and monotonic along the frontiere. For proof see (p. 301-302) of the book.

b) The WE extreme price ratios are determined by the MRS of agents, consuming the whole endowment. The MRS's are continous and monotonic, so the whole range between the two extremes will be achieved along the frontiere.

$MRS_A|_{w=(8,7)} = \frac{2}{y_1} / \frac{1}{y_2}\Big|_{w=(8,7)} = \frac{2y_2}{y_1}\Big|_{y=(8,7)} = \frac{2*7}{8} = \frac{7}{4}$

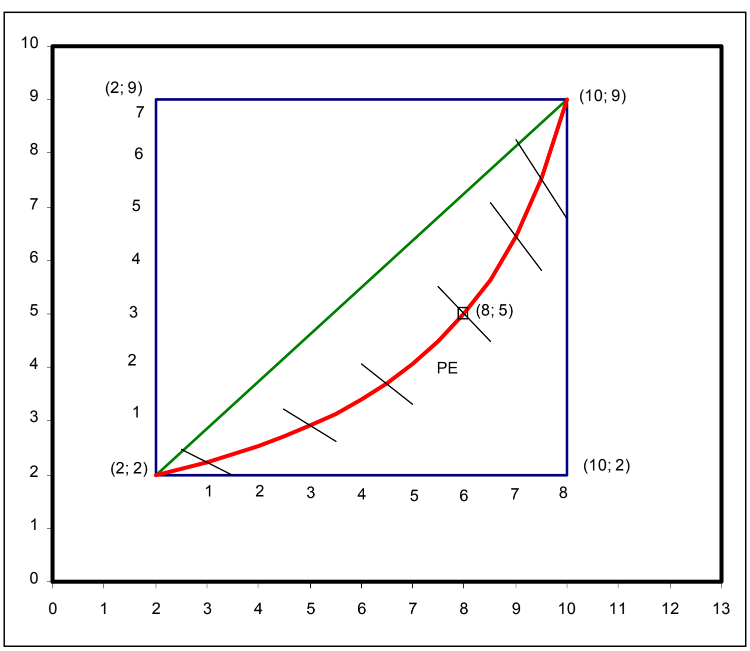
$MRS_B|_{w=(8,7)} = \frac{1}{y_1} / \frac{2}{y_2}\Big|_{w=(8,7)} = \frac{y_2}{2y_1}\Big|_{y=(8,7)} = \frac{7}{2*8} = \frac{7}{16}$

So the price ratios can take values: $\frac{p_1}{p_2} \in \left[\frac{7}{16}, \frac{7}{4}\right]$

c) $w_A = (8,5)$ and $w_B = (5,5)$ corresponds to $w_y^A = (6,3)$ and $w_y^B = (2,4)$ in the small box.

Check that it's a PE: $MRS_A = \frac{2y_2}{y_1}\Big|_{y=(6,3)} = \frac{2*3}{6} = 1 = \frac{4}{2*2} = \frac{y_2}{2y_1}\Big|_{y=(2,4)} = MRS_B$.

The corresponding price ratio is $\frac{p_1}{p_2} = 1 \in \left[\frac{7}{16}, \frac{7}{4}\right]$.



2. Walrasian Equilibria

a) $\alpha^A = (4,6) \qquad \alpha^B = (7,3) \qquad \Rightarrow \qquad w_y = (13-4-7, 10-6-3) = (2,1)$

Therefore, $MRS_A|_{w=(8,7)} = \left.\frac{2y_2}{y_1}\right|_{y=(2,1)} = 1 \qquad MRS_B|_{w=(8,7)} = \left.\frac{y_2}{2y_1}\right|_{y=(2,1)} = \frac{1}{4}$

Henceforth set $p_2 = 1, \qquad p_1 = p \in \left[\frac{1}{4}, 1\right]$

b) $w_y^A = (8,5) - (4,6) = (4,-1) \qquad w_y^B = (5,5) - (7,3) = (-2,2)$

The agents have Cobb-Douglas preferences in the new coordinates, hence, spend fractions of incomes: $I_A(p) = pw_y^A = 4p - 1 \qquad I_B(p) = pw_y^B = 2 - 2p$

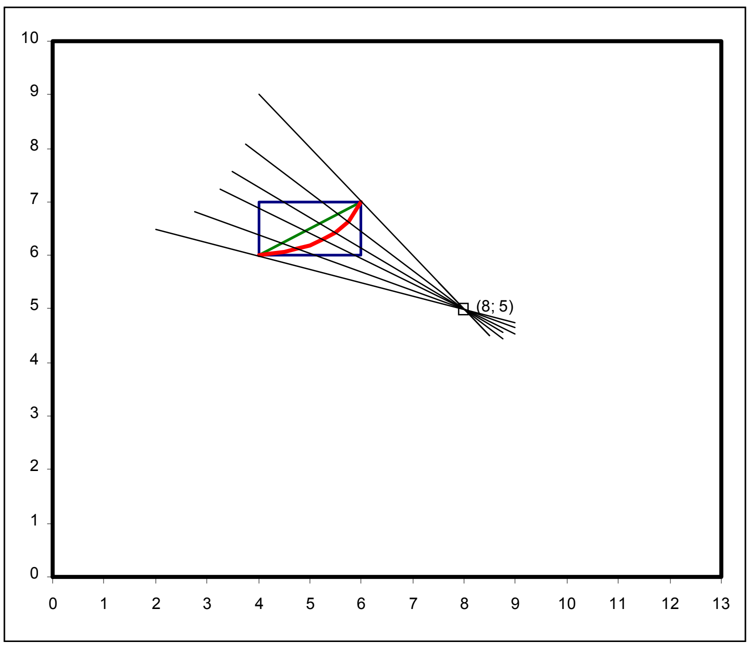
$y_1^A(p) = \frac{2}{3}\frac{I_A(p)}{p} \qquad y_2^A(p) = \frac{1}{3}\frac{I_A(p)}{1} \qquad y_1^B(p) = \frac{1}{3}\frac{I_B(p)}{p} \qquad y_2^A(p) = \frac{2}{3}\frac{I_B(p)}{1}$

Therefore, $e_1(p) = y_1^A(p) + y_1^B(p) - w_{y1}^A - w_{y1}^B = \frac{2}{3}\frac{I_A(p)}{p} + \frac{1}{3}\frac{I_B(p)}{p} - 4 + 2 = \frac{2}{3}\frac{4p-1}{p} + \frac{1}{3}\frac{2-2p}{p} - 4 + 2$

a') Plugging in both $p = \frac{1}{4}$ and $p = 1$ we can see, that $e_1\left(\frac{1}{4}\right) = e_1(1) = 0$.

That means, that both of these are equilibirum prices (by Walras Law the market for commodity 2 is also in equilibrium).

b') Simplifying the expression, we can see, that $e_1(p) = 0$ for all $p \in \left[\frac{1}{4}, 1\right]$. Therefore any price gives an equilibrium. There is a continuum of equilibria.



c) $w_y^A = (10,4) - (4,6) = (6,-2) \qquad w_y^B = (3,6) - (7,3) = (-4,3)$

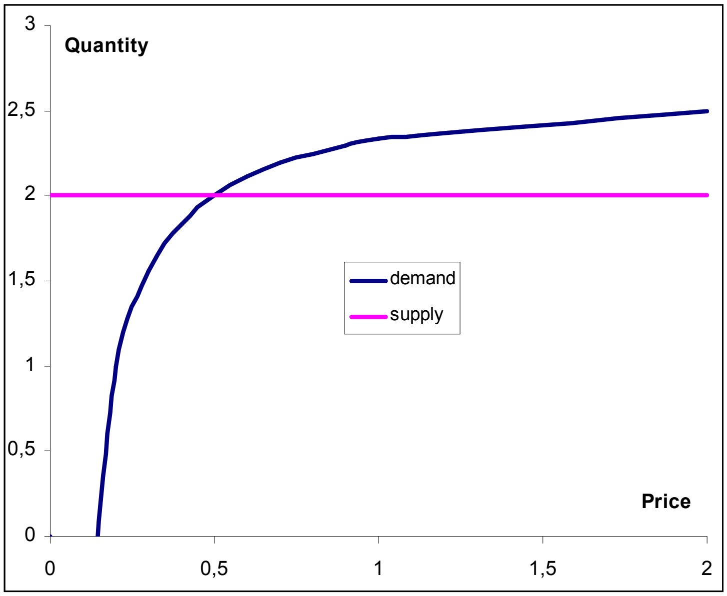
$I_A(p) = pw_y^A = 6p - 2 \qquad I_B(p) = pw_y^B = 3 - 4p$

Therefore, $e_1(p) = \frac{2}{3}\frac{I_A(p)}{p} + \frac{1}{3}\frac{I_B(p)}{p} - 6 + 4 = \frac{2}{3}\frac{6p-2}{p} + \frac{1}{3}\frac{3-4p}{p} - 6 + 4 = \frac{1}{3p}(2p-1)$

$p = \frac{1}{2}$ is the unique solution of the equation: $e_1(p) = 0$. Therefore, it's the only WE price.

c') Demand: $d_1(p) = \frac{2}{3}\frac{6p-2}{p} + \frac{1}{3}\frac{3-4p}{p} = \frac{1}{3p}(8p-1) \qquad$ Supply: $s_1(p) = 6 - 4 = 2$

Supply is inelastic. The demand curve is increasing in price, while usually it is decreasing in price.



3. Production Efficiency

a) $\forall y, z \in Y = Y^1 + Y^2, \exists y^1, z^1 \in Y^1$ and $\exists y^2, z^2 \in Y^2$ s.t. $y = y^1 + y^2$, $z = z^1 + z^2$
Let Y^1 and Y^2 be convex. Now let x be a some convex combination of y and z. Then
$x = \lambda y + (1-\lambda) z = \lambda (y^1 + y^2) + (1-\lambda)(z^1 + z^2) = (\lambda y^1 + (1-\lambda) z^1) + (\lambda y^2 + (1-\lambda) z^2)$
Since Y^1 is convex, $x^1 = (\lambda y^1 + (1-\lambda) z^1) \in Y^1$ - as a convex combination of y^1 and z^1.
Since Y^2 is convex, $x^2 = (\lambda y^2 + (1-\lambda) z^2) \in Y^2$ - as a convex combination of y^2 and z^2.
Therefore, $x = x^1 + x^2$, $x^1 \in Y^1, x^2 \in Y^2 \Rightarrow x \in Y$.
That's true for any convex combination. Hence, Y is convex.
b) Let Y^1 and Y^2 exhibit free disposal. Now let x be the sum of y and z. Then take some $\delta > 0$:
$x - \delta = y - \frac{\delta}{2} + z - \frac{\delta}{2} = x^1 + x^2$
Since Y^1 exhibits free disposal, $x^1 = y - \frac{\delta}{2} \in Y^1$ - belongs to Y^1.
Since Y^2 exhibits free disposal, $x^2 = z - \frac{\delta}{2} \in Y^2$ - belongs to Y^2.
Therefore, $x = x^1 + x^2$, $x^1 \in Y^1, x^2 \in Y^2 \Rightarrow x \in Y$.
That's true for any $\delta > 0$. Hence, Y is also exhibits free disposal.
c) Let $\widehat{y} \in Y$ be efficient, i.e. $\exists p$ s.t. $\forall y \in Y : py \leq p\widehat{y}$
Also, $\exists \widehat{y}^1 \in Y^1$ and $\exists \widehat{y}^2 \in Y^2$ s.t. $\widehat{y} = \widehat{y}^1 + \widehat{y}^2$.
Besides, $\forall y \in Y \quad \exists y^1 \in Y^1$ and $\exists y^2 \in Y^2$ s.t. $y = y^1 + y^2$
If $py^1 > p\widehat{y}^1$ then $p(y^1 + \widehat{y}^2) > p(\widehat{y}^1 + \widehat{y}^2) = p\widehat{y}$
Therefore, since $y^1 \in Y^1, \widehat{y}^2 \in Y^2$, $y\prime = y^1 + \widehat{y}^2 \in Y$.
Hence, $py\prime > p\widehat{y}$, and $\widehat{y}$ is not efficient, which contradicts the assumption.
Therefore, $py^1 \leq p\widehat{y}^1$. By the same logic $py^2 \leq p\widehat{y}^2$.

4. Aggregate Production Sets

$Y^f = \left\{ \left(z^f, q^f\right) | q^f \leq c^f + \alpha^f z^f - \beta^f \left(z^f\right)^2, \alpha^f \geq 0, \beta^f \geq 0 \right\}$

To aggregate the production of the two firms, fix $z = z^1 + z^2$ and find the maximum possible aggregate output.

$Y = Y^1 + Y^2 = \left\{ (z, q) | q^1 \leq c^1 + \alpha^1 z^1 - \beta^1 (z^1)^2, q^2 \leq c^2 + \alpha^2 z^2 - \beta^2 (z^2)^2, z^1 + z^2 = z, q^1 + q^2 = q \right\}$

The constraint could be represented by:

$$q = \max\left\{c^1 + \alpha^1 z^1 - \beta^1 \left(z^1\right)^2 + c^2 + \alpha^2 z^2 - \beta^2 \left(z^2\right)^2 | z_1 + z_2 = z\right\} =$$

$$= \max\left\{c^1 + \alpha^1 z^1 - \beta^1 \left(z^1\right)^2 + c^2 + \alpha^2 \left(z - z^1\right) - \beta^2 \left(z - z^1\right)^2\right\} =$$

FOC: $\alpha^1 - 2\beta^1 z^1 - \alpha^2 + 2\beta^2 \left(z - z^1\right) = 0 \quad \Rightarrow \quad z^1 = \frac{\alpha^1 - \alpha^2 + 2\beta^2 z}{2\left(\beta^1 + \beta^2\right)}, \quad z - z_1 = \frac{\alpha^2 - \alpha^1 + 2\beta^1 z}{2\left(\beta^1 + \beta^2\right)}$

Need to satisfy: $2\beta^2 z \geq \alpha^2 - \alpha^1, \quad 2\beta^1 z \geq \alpha^1 - \alpha^2 \quad \Rightarrow \quad z \geq \max\left\{\frac{\alpha^1 - \alpha^2}{2\beta^1}, \frac{\alpha^2 - \alpha^1}{2\beta^2}, 0\right\}$

$$= \alpha^1 \frac{\alpha^1 - \alpha^2 + 2\beta^2 z}{2\left(\beta^1 + \beta^2\right)} - \beta^1 \left(\frac{\alpha^1 - \alpha^2 + 2\beta^2 z}{2\left(\beta^1 + \beta^2\right)}\right)^2 + \alpha^2 \frac{\alpha^2 - \alpha^1 + 2\beta^1 z}{2\left(\beta^1 + \beta^2\right)} - \beta^2 \left(\frac{\alpha^2 - \alpha^1 + 2\beta^1 z}{2\left(\beta^1 + \beta^2\right)}\right)^2 =$$

$$= \frac{2\left(\beta^1 + \beta^2\right)\left(\alpha^1 - \alpha^2\right)^2 + 4\left(\beta^1 + \beta^2\right)\left(\alpha^1 \beta^2 + \alpha^2 \beta^1\right) z - \beta^1 \left(\alpha^1 - \alpha^2 + 2\beta^2 z\right)^2 - \beta^2 \left(\alpha^2 - \alpha^1 + 2\beta^1 z\right)^2}{4\left(\beta^1 + \beta^2\right)^2} = \frac{\left(\alpha^1 - \alpha^2\right)^2 + 4\left(\alpha^1 \beta^2 + \alpha^2 \beta^1\right) z - 4\beta^1 \beta^2 z^2}{4\left(\beta^1 + \beta^2\right)}$$

So, if $z \geq \max\left\{\frac{\alpha^1 - \alpha^2}{2\beta^1}, \frac{\alpha^2 - \alpha^1}{2\beta^2}, 0\right\}$ then $c' = c^1 + c^2 + \frac{\left(\alpha^1 - \alpha^2\right)^2}{4\left(\beta^1 + \beta^2\right)}, \quad \alpha' = \frac{\alpha^1 \beta^2 + \alpha^2 \beta^1}{\beta^1 + \beta^2}, \quad \beta' = \frac{\beta^1 \beta^2}{\beta^1 + \beta^2}$

If $\alpha^1 > \alpha^2$ and $z < \frac{\alpha^1 - \alpha^2}{2\beta^1} \quad \Rightarrow \quad z_1 = z \quad \Rightarrow \quad q = c^1 + \alpha^1 z - \beta^1 (z)^2 + c^2 \quad \Rightarrow$

$c' = c^1 + c^2, \quad \alpha' = \alpha^1, \quad \beta' = \beta^1$

If $\alpha^2 > \alpha^1$ and $z < \frac{\alpha^2 - \alpha^1}{2\beta^2} \quad \Rightarrow \quad z_1 = 0 \quad \Rightarrow \quad q = c^1 + c^2 + \alpha^2 z - \beta^2 (z)^2 \quad \Rightarrow$

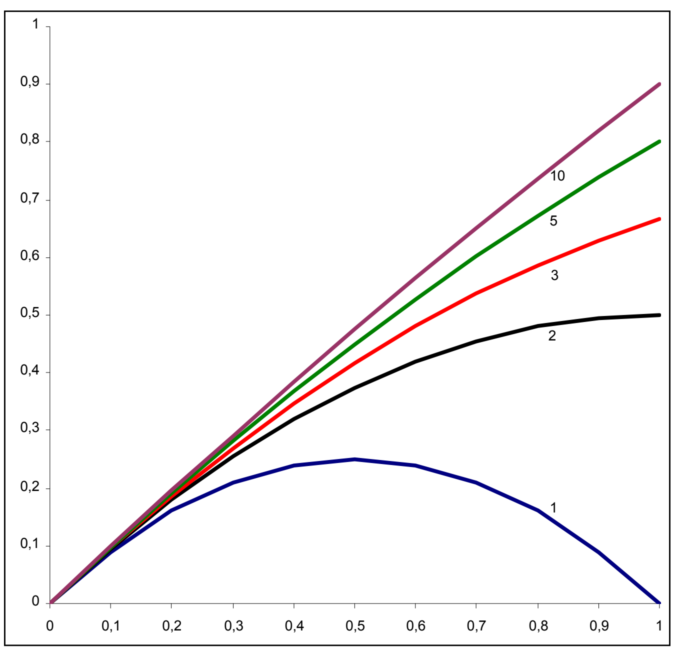
$c' = c^1 + c^2, \quad \alpha' = \alpha^2, \quad \beta' = \beta^2$

The aggregate production set:

$$Y = \left\{ (z, q) \left| \begin{array}{ll} q \leq c^1 + c^2 + \frac{\left(\alpha^1 - \alpha^2\right)^2}{4\left(\beta^1 + \beta^2\right)} + \frac{\alpha^1 \beta^2 + \alpha^2 \beta^1}{\beta^1 + \beta^2} z - \frac{\beta^1 \beta^2}{\beta^1 + \beta^2} (z)^2, & z \geq \max\left\{\frac{\alpha^1 - \alpha^2}{2\beta^1}, \frac{\alpha^2 - \alpha^1}{2\beta^2}, 0\right\} \\ q \leq c^1 + c^2 + \alpha^1 z - \beta^1 (z)^2, & z < \frac{\alpha^1 - \alpha^2}{2\beta^1} > 0 \\ q \leq c^1 + c^2 + \alpha^2 z - \beta^2 (z)^2, & z < \frac{\alpha^2 - \alpha^1}{2\beta^2} > 0 \end{array} \right. \right\}$$

a) $c^1 = c^2 = 0$, $\alpha^1 = \alpha^2 = \alpha$, $\beta^1 = \beta^2 = \beta \Rightarrow \quad c = 0, \quad \alpha' = \frac{\alpha\beta + \alpha\beta}{\beta + \beta} = \alpha, \quad \beta' = \frac{\beta\beta}{\beta + \beta} = \frac{\beta}{2}$

$\Rightarrow \quad Y = \left\{(z, q) | q \leq \alpha z - \frac{\beta}{2} (z)^2\right\}$



b) For n identical firms, we shall recursively aggregate them:

$c_i = c + c_{i-1} + \frac{\left(\alpha_{i-1} - \alpha\right)^2}{4\left(\beta_{i-1} + \beta\right)} \quad \alpha_i = \frac{\alpha_{i-1}\beta + \alpha\beta_{i-1}}{\beta + \beta_{i-1}} \quad \beta_i = \frac{\beta\beta_{i-1}}{\beta + \beta_{i-1}}$

$\beta_1 = \beta \qquad \beta_2 = \frac{\beta\beta}{\beta+\beta} = \frac{\beta}{2} \qquad \beta_3 = \frac{\beta\frac{\beta}{2}}{\beta+\frac{\beta}{2}} = \frac{\beta}{3} \qquad \ldots \qquad \beta_n = \frac{\beta}{n}$

$\alpha_1 = \alpha \qquad \alpha_2 = \frac{\alpha\beta+\alpha\beta}{\beta+\beta} = \alpha \qquad \alpha_3 = \frac{\alpha\beta+\alpha\beta_{i-1}}{\beta+\beta_{i-1}} = \alpha \qquad \ldots \qquad \alpha_n = \alpha$

$c_i = c + c_{i-1} + \frac{(\alpha_{i-1}-\alpha)^2}{4(\beta_{i-1}+\beta)} = c + c_{i-1} = c_{i-1} = 0$ Therefore, $Y_n = \left\{(z,q)\,|q \le \alpha z - \frac{\beta}{n}(z)^2\right\}$

As n grows the term $\widetilde{Y} = \lim_{n\to\infty} Y_n = \left\{(z,q)\,|q \le \alpha z - \lim_{n\to\infty}\frac{\beta}{n}(z)^2\right\} = \{(z,q)\,|q \le \alpha z\}$

Aggregating decreasing-return-to-scale technologies, we got a CRS production set.

c) $\pi_n(q,p,r) = \max\left\{pq - rz|q \le \alpha z - \frac{\beta}{n}z^2\right\} = \max\left\{p\alpha z - p\frac{\beta}{n}z^2 - rz\right\} =$

FOC: $2\frac{\beta}{n}z = \alpha - \frac{r}{p} \qquad q = \left(\alpha - \frac{\beta}{n}z\right)z = \frac{n}{4\beta}\left(\alpha^2 - \left(\frac{r}{p}\right)^2\right)$

$$= \frac{n}{2\beta}\left(\frac{p}{2}\left(\alpha^2 - \frac{r^2}{p^2}\right) - r\left(\alpha - \frac{r}{p}\right)\right) = \begin{cases} 0 & p/r \le 1/\alpha \\ \frac{n}{2\beta}\frac{p}{2}\left(\alpha - \frac{r}{p}\right)^2 & p/r > 1/\alpha \end{cases}$$

$$q_n(p/r) = \begin{cases} 0 & p/r \le 1/\alpha \\ \frac{n}{4\beta}\left(\alpha^2 - \left(\frac{r}{p}\right)^2\right) & p/r > 1/\alpha \end{cases}$$

$$\pi_\infty(q,p,r) = \max\{pq - rz|q \le \alpha z\} = \max\left[p - \frac{r}{\alpha}\right]q = \begin{cases} 0 & p/r \le 1/\alpha \\ \infty & p/r > 1/\alpha \end{cases}$$

$$q_\infty(p/r) = \arg\max\left[p - \frac{r}{\alpha}\right]q = \begin{cases} 0 & p/r < 1/\alpha \\ \forall & p/r = 1/\alpha \\ \infty & p/r > 1/\alpha \end{cases}$$

d) $\alpha^1 > \alpha^2 > 0 \qquad \beta^1 > 0 = \beta^2 \qquad c = \frac{\left(\alpha^1-\alpha^2\right)^2}{4\beta^1} \qquad \alpha = \frac{\alpha^1\beta^2+\alpha^2\beta^1}{\beta^1+\beta^2} = \alpha^2 \qquad \beta = \frac{\beta^1\beta^2}{\beta^1+\beta^2} = 0$

The aggregate production set is: $Y = \left\{(z,q)\;\middle|\; \begin{array}{ll} q \le \frac{\left(\alpha^1-\alpha^2\right)^2}{4\beta^1} + \alpha^2 z & z \ge \frac{\alpha^1-\alpha^2}{2\beta^1} \\ q \le \alpha^1 z - \beta^1(z)^2 & z < \frac{\alpha^1-\alpha^2}{2\beta^1} \end{array}\right\}$

e) The previous example is exactly the case, when the second firm is an aggregate of many identical firms. In this case the production set coincides with the set of the firm for small amounts of input, but becomes linear for large amounts of inputs.