Economics 201A, Homework 2 Answer Key — Anton Cheremukhin

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1. **Production and Cost**

$q = \left(z_1 + 2(z_2)^{1/2}\right)^2 \qquad r = (4,2) \qquad C(Q) = \min\{rz \,|\, q(z) \geq Q\}$

$\pounds = -r_1 z_1 - r_2 z_2 + \lambda\left(\left(z_1 + 2(z_2)^{1/2}\right)^2 - Q\right)$

FOC: $r_1 = \lambda 2 (q(z))^{1/2} \qquad r_2 = \lambda 2 (z_2)^{-1/2} (q(z))^{1/2} \qquad \Rightarrow \qquad r_1/r_2 = (z_2)^{1/2}$

$z_2 = 4 \qquad z_1 = \sqrt{Q} - 4 \qquad \text{if} \qquad Q > 16$

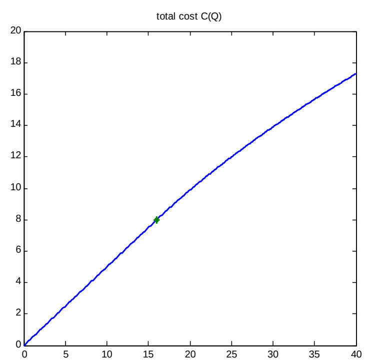
$z_1 = 0 \qquad z_2 = Q/4 \qquad \text{if} \qquad Q \leq 16$

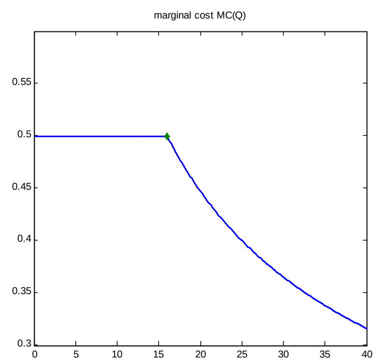
Summing up, $z_1 = \max\left\{\sqrt{Q} - 4, 0\right\} \qquad z_2 = \min\left\{4, \frac{Q}{4}\right\}$

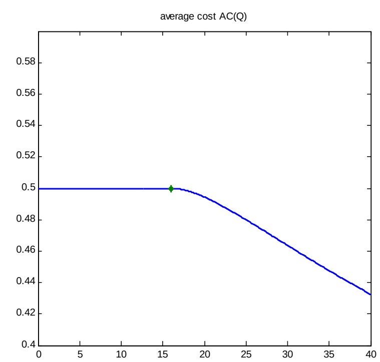
$C(Q) = r_1 z_1 + r_2 z_2 = 4\max\left\{\sqrt{Q} - 4, 0\right\} + 2\min\left\{4, \frac{Q}{4}\right\} = \begin{cases} 4\sqrt{Q} - 8 & , Q > 16 \\ \frac{Q}{2} & , Q \leq 16 \end{cases}$

$MC = \partial C(Q)/\partial Q = \begin{cases} 2/\sqrt{Q} & , Q > 16 \\ 1/2 & , Q \leq 16 \end{cases} \qquad AC = C(Q)/Q = \begin{cases} 4/\sqrt{Q} - 8/Q & , Q > 16 \\ 1/2 & , Q \leq 16 \end{cases}$

The following graphs depict the three curves:







2. **CES and CES like cost functions**

$q = \left((a_1 z_1)^{\frac{\sigma-1}{\sigma}} + (a_2 z_2)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \qquad C(Q) = \min\{rz \,|\, q(z) \geq Q\} \qquad \text{define } y_i = a_i z_i$

a) $q^{\frac{\sigma-1}{\sigma}} = (a_1 z_1)^{\frac{\sigma-1}{\sigma}} + (a_2 z_2)^{\frac{\sigma-1}{\sigma}} \qquad z_2 = \frac{1}{a_2}\left(q^{\frac{\sigma-1}{\sigma}} - (a_1 z_1)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$

Think of $0 < z_1 < q/a_1$ when $\sigma > 1$ and $z_1 > q/a_1$ when $0 < \sigma < 1$.

$\frac{\partial z_2}{\partial z_1} = -\frac{a_1}{a_2 (a_1 z_1)^{\frac{1}{\sigma}}}\left(q^{\frac{\sigma-1}{\sigma}} - (a_1 z_1)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} = -\frac{a_1 (a_2 z_2)^{\frac{1}{\sigma}}}{a_2 (a_1 z_1)^{\frac{1}{\sigma}}} < 0$

$\frac{\partial^2 z_2}{\partial z_1^2} = \frac{q}{\sigma}\frac{a_1}{a_2 z_1}\frac{1}{q^{\frac{1}{\sigma}} (a_1 z_1)^{\frac{1}{\sigma}}}\left(q^{\frac{\sigma-1}{\sigma}} - (a_1 z_1)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{2-\sigma}{\sigma-1}} = \frac{1}{\sigma}\frac{a_1}{a_2}\frac{1}{z_1}\frac{q^{\frac{\sigma-1}{\sigma}}}{(a_2 z_2)^{\frac{\sigma-1}{\sigma}}}\frac{(a_2 z_2)^{\frac{1}{\sigma}}}{(a_1 z_1)^{\frac{1}{\sigma}}} > 0$

So, the upper-contoursets are strictly convex, and the production function is strictly quasi-concave.

b) $C(Q) = \min\left\{\Sigma r_i z_i \middle| \left(\Sigma (a_i z_i)^{1-1/\sigma}\right)^{1/(1-1/\sigma)} \geq Q\right\} = \min\left\{\Sigma \frac{r_i}{a_i} y_i \middle| \left(\Sigma y_i^{1-1/\sigma}\right)^{1/(1-1/\sigma)} \geq Q\right\}$

Lagrangian: $\pounds = -\frac{r_1}{a_1}y_1 - \frac{r_2}{a_2}y_2 + \lambda\left(\left(y_1^{1-1/\sigma} + y_2^{1-1/\sigma}\right)^{1/(1-1/\sigma)} - Q\right)$

FOC: $\frac{r_1}{a_1} = \lambda (y_1)^{-1/\sigma} (q(y))^{1/\sigma} \qquad \frac{r_2}{a_2} = \lambda (y_2)^{-1/\sigma} (q(y))^{1/\sigma} \qquad \Rightarrow \qquad a_2 r_1 / a_1 r_2 = (y_2/y_1)^{1/\sigma}$

$Q = q(y) = \left((y_1)^{1-1/\sigma} + (y_2)^{1-1/\sigma}\right)^{1/(1-1/\sigma)} = y_1\left(1 + (y_2/y_1)^{1-1/\sigma}\right)^{1/(1-1/\sigma)} =$

$= y_1\left(1 + \left(\frac{a_2 r_1}{a_1 r_2}\right)^{\sigma-1}\right)^{1/(1-1/\sigma)} \qquad C(Q) = \frac{r_1}{a_1}y_1 + \frac{r_2}{a_2}y_2 = y_1\frac{r_1}{a_1}\left(1 + \frac{r_2 a_1}{a_2 r_1}\left(\frac{a_2 r_1}{a_1 r_2}\right)^{\sigma}\right) =$

$= y_1\frac{r_1}{a_1}\left(1 + \left(\frac{a_2 r_1}{a_1 r_2}\right)^{\sigma-1}\right) = Q\frac{r_1}{a_1}\left(1 + \left(\frac{a_2 r_1}{a_1 r_2}\right)^{\sigma-1}\right)^{-\frac{1}{\sigma-1}} = Q\frac{\frac{r_1}{a_1}\frac{r_2}{a_2}}{\left(\left(\frac{r_2}{a_2}\right)^{\sigma-1} + \left(\frac{r_1}{a_1}\right)^{\sigma-1}\right)^{\frac{1}{\sigma-1}}}$

c) Define $z_4 = \left((a_1 z_1)^{1-1/\sigma} + (a_2 z_2)^{1-1/\sigma}\right)^{1/(1-1/\sigma)}$ and set $a_1 = a_2 = 1$, $\sigma = 1/2$

We know from (a) that the cost of producing z_4 is

$C(z_4) = z_4 r_1 r_2 \left(r_2^{-1/2} + r_1^{-1/2}\right)^2 = z_4\left(r_2^{1/2} + r_1^{1/2}\right)^2 = z_4 r_4$

Then, $q(z_3, z_4) = z_3 z_4 \qquad C(Q) = \min\{rz \mid q(z) \geq Q\} \qquad \pounds = r_3 z_3 + r_4 z_4 + \lambda(z_3 z_4 - Q)$

FOC: $r_3 = \lambda z_4 \qquad r_4 = \lambda z_3 \qquad \Rightarrow \qquad r_3 z_3 = r_4 z_4$

$Q = z_3 z_4 = (z_3)^2 \frac{r_3}{r_4} \qquad C(Q) = r_3 z_3 + r_4 z_4 = 2 r_3 z_3 = 2 r_3 \left(Q\frac{r_4}{r_3}\right)^{1/2} = 2\sqrt{Q r_3 r_4}$

Therefore, $C(Q) = 2\sqrt{Q}\sqrt{r_3}\left(\sqrt{r_1} + \sqrt{r_2}\right)$

3. Equilibrium and Time in an economy with no production

a) $U(x) = \ln x_1 + 2\ln x_2 + \delta \ln x_3 + 2\delta \ln x_4$

Preferences are homothetic if and only if $\forall x \forall \lambda > 0 \exists \mu > 0 : \nabla U(\lambda x) = \mu \nabla U(x)$

$$\nabla U(x) = \begin{bmatrix} \frac{1}{x_1} \\ \frac{2}{x_2} \\ \frac{\delta}{x_3} \\ \frac{2\delta}{x_4} \end{bmatrix} \qquad \nabla U(\lambda x) = \begin{bmatrix} \frac{1}{\lambda x_1} \\ \frac{2}{\lambda x_2} \\ \frac{\delta}{\lambda x_3} \\ \frac{2\delta}{\lambda x_4} \end{bmatrix} = \frac{1}{\lambda}\begin{bmatrix} \frac{1}{x_1} \\ \frac{2}{x_2} \\ \frac{\delta}{x_3} \\ \frac{2\delta}{x_4} \end{bmatrix} = \frac{1}{\lambda}\nabla U(x) \Rightarrow \mu = \frac{1}{\lambda}$$

I.e. the slope of the indifference curves along a ray coming from the origin is the same.

So, these preferences are homothetic. The same logic works for any T.

b) For agents with identical homothetic preferences (but possibly different endowments), aggregate excess demand can be derived from maximization of a utility function of a representative agent whose endowment is the sum of individual endowments. Such an economy has unique equilibrium, in which each agent consumes a fraction of the aggregate endowment. Endowment in general: $w = (w_1, w_2, w_3, w_4)$. Taking p_1 as the numeraire use the FOCs: $\frac{\partial U(x)/\partial x_i}{\partial U(x)/\partial x_j} = \frac{p_i}{p_j}$

So $p_2 = \frac{\partial U(x)/\partial x_2}{\partial U(x)/\partial x_1} = \frac{2w_1}{w_2} = p_2^s, \qquad p_3 = \frac{\partial U(x)/\partial x_3}{\partial U(x)/\partial x_1} = \frac{\delta w_1}{w_3} = \frac{p_3^s}{1+r} = \frac{1}{1+r}$

$p_4 = \frac{\partial U(x)/\partial x_4}{\partial U(x)/\partial x_1} = \frac{2\delta w_1}{w_4} = \frac{p_4^s}{1+r}$

Hence, $p = (1, \frac{2w_1}{w_2}, \frac{\delta w_1}{w_3}, \frac{2\delta w_1}{w_4})$, $r = \frac{w_3}{\delta w_1} - 1$, $p^s = \left(1, \frac{2w_1}{w_2}, 1, \frac{2w_3}{w_4}\right)$

c) E.g. $w = (10, 20, 6, 12)$, $\delta = 4/5$

$$p = (1, \frac{2w_1}{w_2}, \frac{\delta w_1}{w_3}, \frac{2\delta w_1}{w_4})\Big|_{w_1=10, w_2=20, w_3=6, w_4=12, \delta=4/5} = \left(1, 1, \frac{5}{3}\delta, \frac{5}{3}\delta\right) = \left(1, 1, \frac{4}{3}, \frac{4}{3}\right)$$

$$r = \frac{w_3}{\delta w_1} - 1\Big|_{w_1=10, w_2=20, w_3=6, w_4=12, \delta=4/5} = \frac{3}{5\delta} - 1 = -\frac{1}{4}$$

$$p^s = \left(1, \frac{2w_1}{w_2}, 1, \frac{2w_3}{w_4}\right)\Big|_{w_1=10, w_2=20, w_3=6, w_4=12} = (1, 1, 1, 1)$$

d) E.g. $w = (10, 20, 6, 6)$, $\delta = 4/5$

$p = (1, \frac{2w_1}{w_2}, \frac{\delta w_1}{w_3}, \frac{2\delta w_1}{w_4})\Big|_{w_1=10, w_2=20, w_3=6, w_4=6, \delta=4/5} = \left(1, 1, \frac{5}{3}\delta, \frac{10}{3}\delta\right) = \left(1, 1, \frac{4}{3}, \frac{8}{3}\right)$

$r = \frac{w_3}{\delta w_1} - 1\Big|_{w_1=10, w_2=20, w_3=6, w_4=6, \delta=4/5} = \frac{3}{5\delta} - 1 = -\frac{1}{4}$

$p^s = \left(1, \frac{2w_1}{w_2}, 1, \frac{2w_3}{w_4}\right)\Big|_{w_1=10, w_2=20, w_3=6, w_4=6, \delta=4/5} = (1, 1, 1, 2)$

f) Imagine we have 10 apples today and 6 apples tomorrow. The interesting thing is that we can store them for tomorrow, but can't eat tomorrow's fruit today. To store apples we need MU_1 to be less than MU_3, denoting MU_i the marginal utility of x_j. So, $MU_1 = \frac{1}{w_1} < \frac{\delta}{w_3} = MU_3$ is the condition under which agents would like to store. For part (b) given the initial endowments the condition is equivalent to $\frac{1}{10} < \frac{4/5}{6}$,which is a true statement. The agents will increase their endowment of apples tomorrow at the expense of apples today, untill these marginal utilities are equalized. So the equilibrium condition is the following: $\frac{1}{w_1} = \frac{\delta}{w_3}$, s.t. $w_1 + w_3 = 10 + 6$.

The solution to this system of equations is: $w_1 = 16/(1+\delta) = \frac{80}{9}$, $w_3 = 16\delta/(1+\delta) = \frac{64}{9}$.

The agent will store $\frac{64}{9} - 6 = \frac{10}{9}$ units of commodity 1 to the next period.

g) So, the new endowment vector is: $w = \left(\frac{80}{9}, 20, \frac{64}{9}, 12\right)$

$p = (1, \frac{2w_1}{w_2}, \frac{\delta w_1}{w_3}, \frac{2\delta w_1}{w_4})\Big|_{w_1=\frac{80}{9}, w_2=20, w_3=\frac{64}{9}, w_4=12, \delta=4/5} = \left(1, \frac{8}{9}, 1, \frac{32}{27}\right)$

h) $r = \frac{w_3}{\delta w_1} - 1\Big|_{w_1=\frac{80}{9}, w_2=20, w_3=\frac{64}{9}, w_4=12, \delta=4/5} = 0$

$p^s = \left(1, \frac{2w_1}{w_2}, 1, \frac{2w_3}{w_4}\right)\Big|_{w_1=\frac{80}{9}, w_2=20, w_3=\frac{64}{9}, w_4=12, \delta=4/5} = \left(1, \frac{8}{9}, 1, \frac{32}{27}\right)$

The zero interest rate is a natural result, since today's price of apples has to be equal to todays futures price of apples tomorrow. If there is storage, the agents need to be indifferent between storing and not storing on the margin.

i) $U(x) = \ln x_1 + 2\ln x_2 + \delta \ln x_3 + 2\delta \ln x_4 + \delta^2 \ln x_5 + 2\delta^2 \ln x_6$

Taking p_1 as the numeraire use the FOCs: $\frac{\partial U(x)/\partial x_i}{\partial U(x)/\partial x_j} = \frac{p_i}{p_j}$

So $p_2 = \frac{\partial U(x)/\partial x_2}{\partial U(x)/\partial x_1} = \frac{2w_1}{w_2} = p_2^s$, $\quad p_3 = \frac{\partial U(x)/\partial x_3}{\partial U(x)/\partial x_1} = \frac{\delta w_1}{w_3} = \frac{p_3^s}{1+r_1} = \frac{1}{1+r_1}$

$p_4 = \frac{\partial U(x)/\partial x_4}{\partial U(x)/\partial x_1} = \frac{2\delta w_1}{w_4} = \frac{p_4^s}{1+r_1}$ $\quad p_5 = \frac{\partial U(x)/\partial x_5}{\partial U(x)/\partial x_1} = \frac{\delta^2 w_1}{w_5} = \frac{p_5^s}{(1+r_1)(1+r_2)} = \frac{1}{(1+r_1)(1+r_2)}$

$p_5 = \frac{\partial U(x)/\partial x_5}{\partial U(x)/\partial x_1} = \frac{2\delta^2 w_1}{w_6} = \frac{p_6^s}{(1+r_1)(1+r_2)}$ $\quad$ Hence, $p = (1, \frac{2w_1}{w_2}, \frac{\delta w_1}{w_3}, \frac{2\delta w_1}{w_4}, \frac{\delta^2 w_1}{w_5}, \frac{2\delta^2 w_1}{w_6})$,

$r_1 = \frac{w_3}{\delta w_1} - 1, \quad r_2 = \frac{w_5}{\delta w_3} - 1, \quad p^s = \left(1, \frac{2w_1}{w_2}, 1, \frac{2w_3}{w_4}, 1, \frac{2w_5}{w_6}\right)$

E.g. $w = (10, 20, 6, 12, 5, 10)$

$p = (1, \frac{2w_1}{w_2}, \frac{\delta w_1}{w_3}, \frac{2\delta w_1}{w_4}, \frac{\delta^2 w_1}{w_5}, \frac{2\delta^2 w_1}{w_6})\Big|_{w_1=10, w_2=20, w_3=6, w_4=12, w_5=5, w_6=10, \delta=4/5} = \left(1, 1, \frac{4}{3}, \frac{4}{3}, \frac{32}{25}, \frac{32}{25}\right)$

j) $r_1 = \frac{w_3}{\delta w_1} - 1\Big|_{w_1=10, w_2=20, w_3=6, w_4=12, w_5=5, w_6=10, \delta=4/5} = -\frac{1}{4}$

$r_2 = \frac{w_5}{\delta w_3} - 1\Big|_{w_1=10, w_2=20, w_3=6, w_4=12, w_5=5, w_6=10, \delta=4/5} = \frac{1}{24}$

$p^s = \left(1, \frac{2w_1}{w_2}, 1, \frac{2w_3}{w_4}, 1, \frac{2w_5}{w_6}\right)\Big|_{w_1=10, w_2=20, w_3=6, w_4=12, w_5=5, w_6=10, \delta=4/5} = (1, 1, 1, 1, 1, 1)$