Economics 201A, Homework 1 Answer Key Anton Cheremukhin

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1. **Modified Cobb-Douglas Consumers**

First, we make a positive monotonic transformation generalizing for n goods:

$$\ln U(x) = \sum_{i=1}^{n} \alpha_i \ln(x_i + b_i) \rightarrow \max_x \qquad s.t. \qquad p \cdot x \le I$$

Then we define $y_j \equiv x_j + b_j$ and restate the problem:

$$\ln U(y) = \sum_{i=1}^{n} \alpha_i \ln y_i \rightarrow \max_y \qquad s.t. \qquad p \cdot y \le I + p \cdot b$$

This is a standard Cobb-Douglas problem for which we know the solution:

$$p_j y_j = \frac{\alpha_j}{\Sigma_{i=1}^{n} \alpha_i} (I + p \cdot b)$$

a) Assuming positive demand, this implies the following demand functions:

$$x_j = \frac{\alpha_j}{\Sigma_{i=1}^{n} \alpha_i} \frac{(I + p \cdot b)}{p_j} - b_j$$

b) Therefore, for the demand vector to be strictly positive we need:

$$\frac{\alpha_j}{\Sigma_{i=1}^{n} \alpha_i} (I + p \cdot b) > b_j p_j$$

It is a sufficient condition, because the utility is strictly monotonic and strictly concave.

The corresponding parameter vector θ is $\left(\frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3}, \frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_3}, \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} \right)$

c) Assuming $\alpha = p = (1,1,1)$ and $b = (3,4,5)$, we are asked to find the income levels, for which only one good will be consumed. As we derived previously, the demand functions are the following:

$$x = \left\{ \frac{1}{3} \frac{(I + (3+4+5))}{1} - b_j \right\}_{j=1,2,3} = \left\{ \frac{1}{3} I - b_j + 4 \right\}_{j=1,2,3} = \left\{ \begin{array}{c} \frac{1}{3} I + 1 \\ \frac{1}{3} I \\ \frac{1}{3} I - 1 \end{array} \right.$$

It is clear, that for $\frac{1}{3} I - 1 \le 0 \quad \Leftrightarrow \quad I \le 3$ the third good is not consumed.

Because our function is defined on the axis, but negative consumption is prohibited, for lower values of income consumption of good three will be exactly equal to zero.

Given that, the problem simplifies to:

$$\ln U(x) = \sum_{i=1}^{2} \alpha_i \ln(x_i + b_i) + \alpha_3 \ln(b_3) \rightarrow \max_x \qquad s.t. \qquad p \cdot x \le I$$

So, basically, we have the same problem but for two goods. It's solution we have already derived.

$$x = \left\{ \frac{1}{2} \frac{(I + (3+4))}{1} - b_j \right\}_{j=1,2} = \left\{ \frac{1}{2} I - b_j + \frac{7}{2} \right\}_{j=1,2} = \left\{ \begin{array}{c} \frac{1}{2} I + \frac{1}{2} \\ \frac{1}{2} I - \frac{1}{2} \end{array} \right.$$

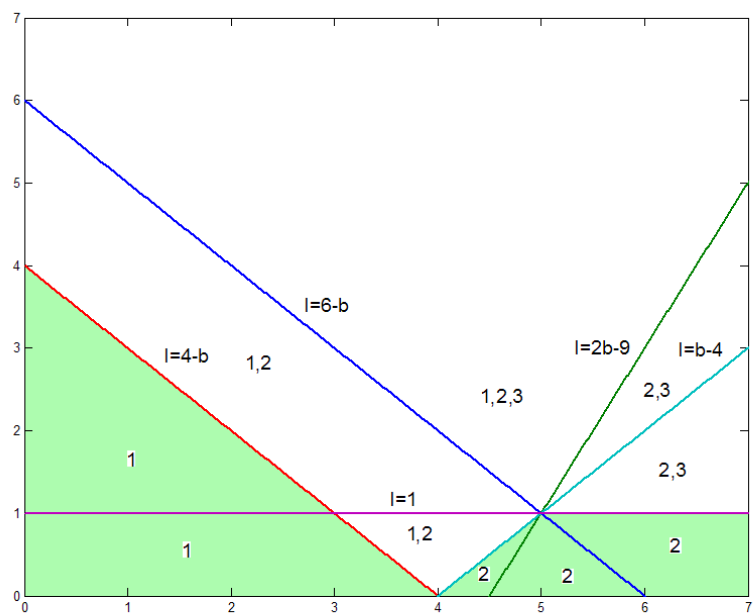
Again, it is clear that for $I \le 1$ the second good is not consumed.

So, when $(I \le 1) \cap (I \le 3) = (I \le 1)$ both goods 2 and 3 are not consumed. Answer: $\boxed{I \le 1}$.

d) Assuming $\alpha = p = (1,1,1)$ and $b = (b_1, 4, 5)$, we repeat the same logic:

$$x = \left\{ \frac{1}{3} \frac{(I + (b_1+4+5))}{1} - b_j \right\}_{j=1,2,3} = \left\{ \frac{1}{3} I + \frac{1}{3} b_1 - b_j + 3 \right\}_{j=1,2,3} = \left\{ \begin{array}{c} \frac{1}{3} I - \frac{2}{3} b_1 + 3 \\ \frac{1}{3} I + \frac{1}{3} b_1 - 1 \\ \frac{1}{3} I + \frac{1}{3} b_1 - 2 \end{array} \right.$$

For $(I \leq 6 - b_1) \cap (I \geq 2b_1 - 9)$ the third good is not consumed.
For $(I \geq 6 - b_1) \cap (I \leq 2b_1 - 9)$ the first good is not consumed.
For the case $(I + b_1 - 6 \leq 0) \cap (I - 2b_1 + 9 \leq 0)$ we have to check both subcases.
First consider the case when good three is not consumed. Then the choice is determined by:
$x = \left\{ \frac{1}{2}\frac{(I+(b_1+4))}{1} - b_j \right\}_{j=1,2} = \left\{ \frac{1}{2}I + \frac{1}{2}b_1 - b_j + 2 \right\}_{j=1,2} = \begin{cases} \frac{1}{2}I - \frac{1}{2}b_1 + 2 \\ \frac{1}{2}I + \frac{1}{2}b_1 - 2 \end{cases}$
For $(I \leq 4 - b_1) \cap (I \geq b_1 - 4)$ the second good is not consumed.
For $(I \leq b_1 - 4) \cap (I \geq 4 - b_1)$ the first good is not consumed.
The case $(I \leq b_1 - 4) \cap (I \leq 4 - b_1)$ is only possible when $I = 0, b_1 = 4$.
Next consider the case when good one is not consumed. Then the choice is determined by:
$x = \left\{ \frac{1}{2}\frac{(I+(4+5))}{1} - b_j \right\}_{j=2,3} = \left\{ \frac{1}{2}I - b_j + \frac{9}{2} \right\}_{j=2,3} = \begin{cases} \frac{1}{2}I + \frac{1}{2} \\ \frac{1}{2}I - \frac{1}{2} \end{cases}$
Clearly, only if $I \leq 1$ is it that good three won't be consumed.
Now, let's intersect the cases with the subcases:
Only good 1 is consumed if $(I \leq 6 - b_1) \cap (I \geq 2b_1 - 9) \cap (I \leq 4 - b_1) \cap (I \geq b_1 - 4)$.
This is equivalent to $\boxed{I \leq 4 - b_1}$.
Only good 2 is consumed if $(I \leq 6 - b_1) \cap (I \leq b_1 - 4) \cap (I \geq 4 - b_1)$
or $(I \geq 6 - b_1) \cap (I \leq 1)$ which is equivalent to $\boxed{(I \leq b_1 - 4) \cap (I \leq 1)}$.
Good 3 will never be consumed alone.
The graph summarizes the regions of interest.



It suggests, that for sufficiently low incomes for all values of $b_1 \neq 4$ only one good is consumed.

2. **Which Commodities?**

$U = x_1^\alpha \left(x_2^\beta + x_3^\beta\right) \to \max_x \quad s.t. \quad p \cdot x \le I$

a) When $\beta = 1$ it simplifies to: $U = x_1^\alpha (x_2 + x_3) \to \max_x \quad s.t. \quad p \cdot x \le I$

Goods 2 and 3 are perfect substitutes, we can denote them $y = x_2 + x_3$.

We get : $U = x^\alpha y \to \max_{x,y} \quad s.t. \quad (p_x, p_y) \cdot (x, y) \le I$

The demand in this setting is standart: $p_x x = \frac{\alpha}{\alpha+1} I, \quad p_y y = \frac{1}{\alpha+1} I.$

Now, the distribution of y between x_2 and x_3 depends on their relative prices.

In our case, we shall consume the one which is cheaper:

$$(x_2, x_3) = \begin{cases} (y, 0) & p_2 < p_3 \\ (t, y - t) & p_2 = p_3 \\ (0, y) & p_2 > p_3 \end{cases}, \text{ where } t \in [0, y]. \qquad \text{It's also clear that price } p_x = \min[p_2, p_3].$$

Therefore, $y = \frac{1}{\alpha+1} \frac{I}{\min[p_2,p_3]}$. Consumption of good 1 stays the same. Summarizing, we get:

$$(x_1, x_2, x_3) = \begin{cases} \left(\frac{\alpha}{\alpha+1}\frac{I}{p_1}, \frac{1}{\alpha+1}\frac{I}{\min[p_2,p_3]}, 0\right) & p_2 < p_3 \\ \left(\frac{\alpha}{\alpha+1}\frac{I}{p_1}, t, \frac{1}{\alpha+1}\frac{I}{\min[p_2,p_3]} - t\right) & p_2 = p_3 \\ \left(\frac{\alpha}{\alpha+1}\frac{I}{p_1}, 0, \frac{1}{\alpha+1}\frac{I}{\min[p_2,p_3]}\right) & p_2 > p_3 \end{cases}, \qquad \text{where } t \in \left[0, \frac{1}{\alpha+1}\frac{I}{\min[p_2,p_3]}\right].$$

b) The logic is basically the same, except for the fact, that the inner function is not concave.

$U = x_1^\alpha (x_2^2 + x_3^2) \to \max_x \quad s.t. \quad p \cdot x \le I$

So, we shall first look at the inner function: $x_2^2 + x_3^2 \to \max_{x_2,x_3} \quad s.t. \ (p_2, p_3) \cdot (x_2, x_3) \le p_y y$

The solution to this problem is always a corner solution, such that the cheaper good is consumed only. The corresponding demand is: $(x_2, x_3) = \begin{cases} (y, 0) & p_2 \le p_3 \\ (0, y) & p_2 \ge p_3 \end{cases}, \quad$ and $p_y = \min[p_2, p_3]$.

So the problem simplifies to: $U = x_1^\alpha x_i^2 \to \max_x \quad s.t. \quad p_1 x_1 + p_i x_i \le I \qquad i = \arg\min[p_2, p_3]$.

This is the again our standard Cobb-Douglas case with: $p_1 x_1 = \frac{\alpha}{\alpha+2} I, \quad p_i x_i = \frac{2}{\alpha+2} I.$

$$\text{Summarizing, we get: } (x_1, x_2, x_3) = \begin{cases} \left(\frac{\alpha}{\alpha+2}\frac{I}{p_1}, \frac{2}{\alpha+2}\frac{I}{\min[p_2,p_3]}, 0\right) & p_2 \le p_3 \\ \left(\frac{\alpha}{\alpha+2}\frac{I}{p_1}, 0, \frac{2}{\alpha+2}\frac{I}{\min[p_2,p_3]}\right) & p_2 \ge p_3 \end{cases}.$$

3. **CES function**

$U = x_1^\rho + x_2^\rho \to \max_x \quad s.t. \quad p \cdot x \le I$

$L = x_1^\rho + x_2^\rho + \lambda (I - p_1 x_1 - p_2 x_2)$

$\text{FOC}_{x_i}: \quad \rho x_i^{\rho-1} = \lambda p_i \quad \Leftrightarrow \quad \rho x_i^\rho = \lambda p_i x_i \qquad \text{BC:} \qquad I = p_1 x_1 + p_2 x_2 \qquad (2)$

Combining the FOC's we get: $\frac{p_2}{p_1} = \left(\frac{x_2}{x_1}\right)^{\rho-1} \qquad (3)$

Plugging (3) into (2) we get: $I = p_1 x_1 + p_2 x_2 = p_1 x_1 \left(1 + \left(\frac{x_2}{x_1}\right)^\rho\right) = p_1 x_1 \left(1 + \left(\frac{p_2}{p_1}\right)^{\frac{\rho}{\rho-1}}\right)$

Therefore, $x_1^* = \frac{I}{p_1 + p_1\left(\frac{p_2}{p_1}\right)^{\frac{\rho}{\rho-1}}} = \frac{I}{p_1 + p_1^{-\frac{1}{\rho-1}} p_2^{\frac{\rho}{\rho-1}}} = \frac{I \cdot p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}$

Similarly, $x_2^* = \frac{I}{p_2 + p_2\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{\rho-1}}} = \frac{I}{p_2 + p_2^{-\frac{1}{\rho-1}} p_1^{\frac{\rho}{\rho-1}}} = \frac{I \cdot p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}$

This implies, that $U(x^*) = x_1^\rho + x_2^\rho = \left(\frac{I \cdot p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}\right)^\rho + \left(\frac{I \cdot p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}\right)^\rho =$

$$= \frac{I^\rho}{\left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)^\rho}\left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right) = \left(\frac{I}{\left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}}\right)^\rho$$

For $\rho = \frac{1}{2}$ we get: $U(x^*) = \sqrt{I\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}$ which accomplishes the proof.

b) Now, $U = \left(x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}\right)x_3^{\frac{1}{2}} \to \max_x \qquad s.t. \qquad p \cdot x \le I$

$\frac{\partial U}{\partial x} = \left[\frac{\sqrt{x_3}}{2\sqrt{x_1}}, \frac{\sqrt{x_3}}{2\sqrt{x_2}}, \frac{\sqrt{x_1}+\sqrt{x_2}}{2\sqrt{x_3}}\right]$ Note, that $\lim_{x_i \to 0} \frac{\partial U}{\partial x_i} = +\infty$.

This condition guarantees, that for convex preferences mixed bundles are always preferred.

$$\frac{\partial^2 U}{\partial x^2} = \begin{bmatrix} -\frac{\sqrt{x_3}}{4x_1^{\frac{3}{2}}} & 0 & \frac{1}{4\sqrt{x_1}\sqrt{x_3}} \\ 0 & -\frac{\sqrt{x_3}}{4x_2^{\frac{3}{2}}} & \frac{1}{4\sqrt{x_2}\sqrt{x_3}} \\ \frac{1}{4\sqrt{x_1}\sqrt{x_3}} & \frac{1}{4\sqrt{x_2}\sqrt{x_3}} & -\frac{(\sqrt{x_1}+\sqrt{x_2})}{4x_3^{\frac{3}{2}}} \end{bmatrix} \qquad -\frac{\sqrt{x_3}}{4x_1^{\frac{3}{2}}} \le 0$$

$$\det\begin{bmatrix} -\frac{\sqrt{x_3}}{4x_1^{\frac{3}{2}}} & 0 \\ 0 & -\frac{\sqrt{x_3}}{4x_2^{\frac{3}{2}}} \end{bmatrix} = \frac{1}{16x_1^{\frac{3}{2}}x_2^{\frac{3}{2}}}x_3 \ge 0 \qquad \det\begin{bmatrix} -\frac{\sqrt{x_3}}{4x_1^{\frac{3}{2}}} & 0 & \frac{1}{4\sqrt{x_1}\sqrt{x_3}} \\ 0 & -\frac{\sqrt{x_3}}{4x_2^{\frac{3}{2}}} & \frac{1}{4\sqrt{x_2}\sqrt{x_3}} \\ \frac{1}{4\sqrt{x_1}\sqrt{x_3}} & \frac{1}{4\sqrt{x_2}\sqrt{x_3}} & -\frac{(\sqrt{x_1}+\sqrt{x_2})}{4x_3^{\frac{3}{2}}} \end{bmatrix} = 0 \le 0$$

So, the matrix is negative semidefinite, and the utility function is concave.
This together with the limit property guarantees unique interior solution.
Now, let's define the amount of money, allocated to the first two goods, as y.
Then the utility of spending y units of money on the first two goods is described by:

$U_{1,2}(x^*) = \sqrt{y\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}$

Therefore, the problem can be reformulated as:

$U = y^{\frac{1}{2}}x_3^{\frac{1}{2}} \to \max_x \qquad s.t. \qquad y + p_3x_3 \le I$ Solving this one we get: $y = \frac{I}{2} = p_3x_3$.

The demand for x_1 and x_2 can be recovered, using part (a).

$$x_1 = \left.\frac{y \cdot p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}}\right|_{\rho=\frac{1}{2}} = \frac{y}{p_1^2\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \qquad x_2 = \left.\frac{y \cdot p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}}\right|_{\rho=\frac{1}{2}} = \frac{y}{p_2^2\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}$$

Therefore, $(x_1, x_2, x_3) = \left(\frac{I/2}{p_1^2\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}, \frac{I/2}{p_2^2\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}, \frac{I/2}{p_3}\right)$.

4. **Exponential Consumer**

$U(x) = -b_1 e^{-Ax_1} - b_2 e^{-Ax_2} \to \max_x \qquad s.t. \qquad p \cdot x \le I$
$L = -b_1 e^{-Ax_1} - b_2 e^{-Ax_2} + \lambda (I - p_1 x_1 - p_2 x_2)$
$\text{FOC}_{x_i}: \qquad -Ab_i e^{-Ax_i} = \lambda p_i \qquad \text{BC}: \qquad I = p_1 x_1 + p_2 x_2$
Combining the FOC's we get: $\quad \frac{p_2}{p_1} = \frac{-Ab_2 e^{-Ax_2}}{-Ab_1 e^{-Ax_1}} = \frac{b_2}{b_1} e^{-A(x_2 - x_1)}$
So, $\qquad x_2 = x_1 - \frac{1}{A} \ln \frac{p_2}{p_1} \frac{b_1}{b_2}.$
$I = p_1 x_1 + p_2 \left(x_1 - \frac{1}{A} \ln \frac{p_2}{p_1} \frac{b_1}{b_2} \right) = (p_1 + p_2) x_1 - \frac{p_2}{A} \ln \frac{b_1 p_2}{b_2 p_1}$
Therefore, $x_1 = \frac{I + \frac{p_2}{A} \ln \frac{b_1 p_2}{b_2 p_1}}{p_1 + p_2} \qquad x_2 = \frac{I + \frac{p_1}{A} \ln \frac{p_1}{p_2} \frac{b_2}{b_1}}{p_1 + p_2}.$
Consumption of commodity 1 will be zero, if $I \le \frac{p_2}{A} \ln \frac{b_2 p_1}{b_1 p_2}$.
Consumption of commodity 2 will be zero, if $I \le \frac{p_1}{A} \ln \frac{b_1 p_2}{b_2 p_1}$.

5. **Pareto Efficiency**

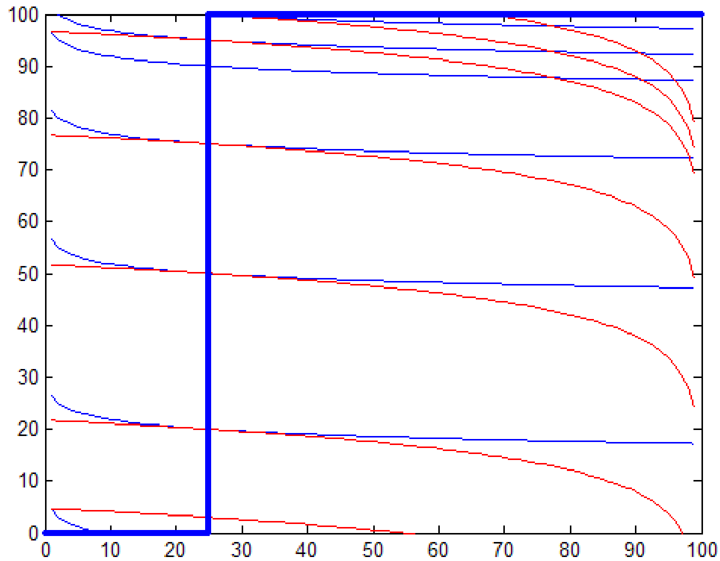
a) Pareto problem: $x_1^\alpha x_2^\beta \to \max_x \qquad s.t. \qquad y_1^\gamma y_2^\delta \ge U$
$s.t. \qquad x_1 + y_1 \le w_1 \qquad s.t. \qquad x_2 + y_2 \le w_2$
$L = x_1^\alpha x_2^\beta + \lambda \left(y_1^\gamma y_2^\delta - U \right) + \mu_1 (w_1 - x_1 - y_1) + \mu_2 (w_2 - x_2 - y_2)$
$\text{FOC}_{x_1}: \frac{\alpha}{x_1} x_1^\alpha x_2^\beta = \mu_1 \qquad \text{FOC}_{x_2}: \frac{\beta}{x_2} x_1^\alpha x_2^\beta = \mu_2$
$\text{FOC}_{y_1}: \frac{\gamma}{y_1} y_1^\gamma y_2^\delta = \mu_1 \qquad \text{FOC}_{y_2}: \frac{\delta}{y_2} y_1^\gamma y_2^\delta = \mu_2$
Combining these we get: $\frac{\alpha}{x_1} x_1^\alpha x_2^\beta = \frac{\gamma}{y_1} y_1^\gamma y_2^\delta, \qquad \frac{\delta}{y_2} y_1^\gamma y_2^\delta = \frac{\beta}{x_2} x_1^\alpha x_2^\beta$
Which implies: $\frac{\alpha}{x_1} / \frac{\beta}{x_2} = \frac{\gamma}{y_1} / \frac{\delta}{y_2}$. Let's use the resourse constraints now: $\quad \frac{\alpha}{\beta} \frac{x_2}{x_1} = \frac{\gamma}{\delta} \frac{y_2}{y_1} = \frac{\gamma}{\delta} \frac{w_2 - x_2}{w_1 - x_1}$
In case $\alpha = \gamma$ and $\beta = \delta$ the solution simplifies to:
$\frac{x_2}{x_1} = \frac{w_2 - x_2}{w_1 - x_1} \qquad \Leftrightarrow \qquad x_2 = \frac{w_2}{w_1} x_1 \qquad$ This is the diagonal of the Edgeworth box.
This is a mere coincidence, which reflects the fact, that both Alex and Bev have identical preferences.

b) $2 \ln x_1^A + x_2^A \to \max_x \qquad s.t. \qquad 6 \ln x_1^B + x_2^B \ge U$
$s.t. \qquad x_1^A + x_1^B \le 100 \qquad s.t. \qquad x_2^A + x_2^B \le 100$
Plugging in the resource constraints, we simplify the Lagrangian:
$L = 2 \ln x_1^A + x_2^A + \lambda \left(6 \ln \left(100 - x_1^A \right) + 100 - x_2^A - U \right)$
$\text{FOC}_{x_1}: \frac{2}{x_1} = \lambda \frac{6}{100 - x_1} \qquad \text{FOC}_{x_2}: 1 = \lambda$
Therefore, the Pareto-optimal frontiere in the interior of the Edgeworth box is determined by:
$\frac{2}{x_1} = \frac{6}{100 - x_1} \qquad \Leftrightarrow \qquad x_1 = 25$

c) Plugging the solution into Bev's utility we get:
$U^B = 6 \ln \left(100 - x_1^A \right) + 100 - x_2^A = 6 \ln 75 + 100 - x_2^A$
In the interior, $x_2^A \in (0, 100)$. $\qquad$ So, $U^B \in (6 \ln 75, 6 \ln 75 + 100)$.

d) Besides the vertical line, we also know, that the Pareto-efficient frontier is a continuous function, and that the origins of both Alex and Bev have to be on it. Therefore, the only possibility is that the edges, connecting the origins and the points we derived, are on the Pareto-efficient line.

The proof of this in this particular case is rather simple. Notice, that the slope of the utility function is independent of the amount of the second good consumed. Therefore, everywhere to the right of the internal PE there is a potential improvement for both A and B if they go left and up. Therefore, only the line, connecting (25,100) and (100,100) can be on the PE besides the vertical segment. A similar argument holds for the case, when we take a point to the left of PE. The graph illustrates the PE frontier and the slopes of the curves.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