Other problems to think of:

2. Input price effects

(a) A firm is a price taker in all markets. Prove the first law of input demand, that is $\Delta r_j \Delta z_j \leq 0$. (4 points)

(b) See if you can extend this result if the firm is a monopolist in all of its output markets. (4 points)

(c) Returning to the case of price taking in all markets, if the price of input 1 rises, under what conditions, if any, will output of commodity 1 rise. Explain. (2 points)

2. Input price effects

(a) Let (q^0, z^0) be optimal for (p, r^0), and (q^1, z^1) be optimal for (p, r^1).

Since profit must be maximized by the optimal choice,

$$p \cdot q^0 - r^0 \cdot z^0 \geq p \cdot q^1 - r^0 \cdot z^1 \ldots (1) \text{ and } p \cdot q^1 - r^1 \cdot z^1 \geq p \cdot q^0 - r^1 \cdot z^0 \ldots (2)$$

By adding both sides of (1) and (2), cancelling out terms and rearranging,

$$r^0 \cdot (z^1 - z^0) - r^1 \cdot (z^1 - z^0) \geq 0 \Longrightarrow (r^1 - r^0) \cdot (z^1 - z^0) \leq 0$$

By setting $r_i^1 = r_i^0$ except for $i = j$, we obtain

$(r_j^1 - r_j^0)(z_j^1 - z_j^0) \leq 0$, i.e. $\Delta r_j \Delta z_j \leq 0$, as we wished to show.

(b) Again, let (q^0, z^0) be optimal for r^0 and (q^1, z^1) be optimal for r^1.

Furthermore, let $R(q^0) \equiv \sum_i p_i(q_i^0) q_i^0$ and $R(q^1) \equiv \sum_i p_i(q_i^1) q_i^1$.

Again, from the profit maximization condition,

$$R(q^0) - r^0 \cdot z^0 \geq R(q^1) - r^0 \cdot z^1 \ldots (3) \text{ and } R(q^1) - r^1 \cdot z^1 \geq R(q^0) - r^1 \cdot z^0 \ldots (4)$$

By adding both sides of (3) and (4), cancelling out terms and rearranging,

$$r^0 \cdot (z^1 - z^0) - r^1 \cdot (z^1 - z^0) \geq 0 \Longrightarrow (r^1 - r^0) \cdot (z^1 - z^0) \leq 0$$

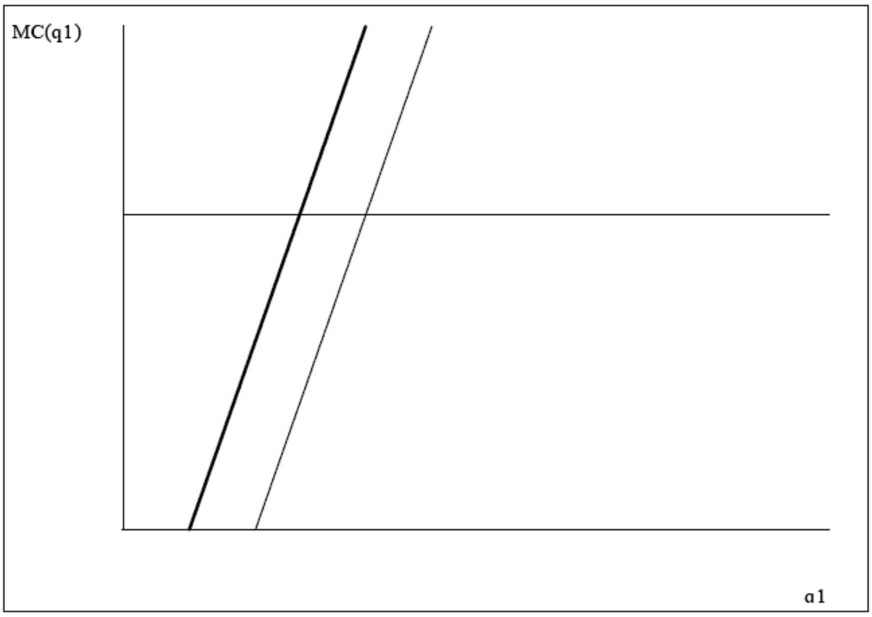
By setting $r_i^1 = r_i^0$ except for $i = j$, again we obtain

$$(r_j^1 - r_j^0)(z_j^1 - z_j^0) \leq 0, \text{ i.e. } \Delta r_j \Delta z_j \leq 0$$

so the first law of input demand holds even when the firm is a monopolist in all of its output market.

(c) Since the firm is a price-taker, it produces the quantity such that $p = MC$.

(In the figure below MC is drawn to be linear in q, but that's just for the sake of simplicity.)



From the figure, we observe that for the output of commodity 1 to rise, its MC must decrease when r_1 rises.

Assuming twice differentiability of the cost function of commodity 1, this implies

$$\frac{\partial MC(q_1)}{\partial r_1} \leq 0 \Longleftrightarrow \frac{\partial^2 C(q_1)}{\partial r_1 \partial q_1} \leq 0$$

$$\Longleftrightarrow \frac{\partial^2 C(q_1)}{\partial q_1 \partial r_1} \leq 0 \quad (\because \text{By Young's theorem, } \frac{\partial^2 C(q_1)}{\partial r_1 \partial q_1} = \frac{\partial^2 C(q_1)}{\partial q_1 \partial r_1})$$

$$\Longleftrightarrow \frac{\partial z_1^*}{\partial q_1} \leq 0 \qquad (\because \text{By the envelope theorem,} \frac{\partial C(q_1)}{\partial r_1} = z_1^*)$$

Hence, the required condition is that the input 1 is an inferior input in the production of commodity 1.

2. Price taking firm (10 points)

A profit maximizing firm is a price taker in all markets. The initial price vector is $p^0 = (p_1^0, \ldots, p_n^0)$. Which of the following are true?
(Derive your answer or explain why the statement is false.)

(a) If p_1 rises the change in output $\Delta y_1 \geq 0$

(b) If p_1 and p_2 both rise, the change in outputs $\Delta y_1 \geq 0$, $\Delta y_2 \geq 0$.

(c) If $p^1 = (\theta p_1^0, \theta p_2^0, p_3^0 \ldots, p_n^0)$ where $\theta > 1$ (the first two prices increase proportionally), then $p_1^0 \Delta y_1 + p_2^0 \Delta y_2 \geq 0$

PART A (6 POINTS)

This statement is true. One way to see it is to say that the profit maximizing firm sets marginal revenue equal to marginal cost. Since the firm is a price taker, marginal revenue is equal to price. When the price increases, marginal cost has not increased but marginal revenue has increased. Therefore, if the marginal cost line is upward sloping, then output increases. If it is flat then output does not change.

A more formal argument would go:

call by $\mathbf{q}^1$ the profit max output choice when prices are $\mathbf{p}^1$

call by $\mathbf{q}^0$ the profit max output choice when prices are $\mathbf{p}^0$

$$\left.\begin{array}{l}\mathbf{p}^1\mathbf{q}^1 \geq \mathbf{p}^0\mathbf{q}^1 \\ -\mathbf{p}^1\mathbf{q}^0 \geq -\mathbf{p}^0\mathbf{q}^0\end{array}\right\} \Rightarrow \mathbf{p}^1(\mathbf{q}^1 - \mathbf{q}^0) \geq \mathbf{p}^0(\mathbf{q}^1 - \mathbf{q}^0) \Rightarrow (\mathbf{p}^1 - \mathbf{p}^0)(\mathbf{q}^1 - \mathbf{q}^0) \geq 0 \qquad \begin{bmatrix} p_1^1 - p_1^0 & \cdots & p_n^1 - p_n^0 \end{bmatrix} \begin{bmatrix} q_1^1 - q_1^0 \\ \vdots \\ q_n^1 - q_n^0 \end{bmatrix} \geq 0$$

which means:

$$\sum_{i=1}^{n} (p_i^1 - p_i^0)(q_i^1 - q_i^0) \geq 0 \qquad (p_1^1 - p_1^0)(q_1^1 - q_1^0) + \sum_{i=2}^{n} (p_i^1 - p_i^0)(q_i^1 - q_i^0) \geq 0$$

Now, since the summation is equal to zero (only the price of good 1 is changing in part A), and since the price difference for good 1 is positive (the price has risen), we know that the change in output must be nonnegative.

PART B (2 POINTS)

This is not necessarily true. Have a look at the expression again:

$$(p_1^1 - p_1^0)(q_1^1 - q_1^0) + (p_2^1 - p_2^0)(q_2^1 - q_2^0) + \sum_{i=3}^{n} (p_i^1 - p_i^0)(q_i^1 - q_i^0) \geq 0$$

Now, even though we know the summation is zero, and the two price differences are positive, we could have for example one change in output that is very positive, and one change which is only a little negative. An intuitive way to say this is that if both prices increase but if the price of good 1 increases say by a great deal more, then the firm might choose to divert some inputs from production of good 2 into production of good 1, thereby greatly increasing output of good 1 while slightly decreasing output of good 2.

PART C (2 POINTS)

This is true. Let's have a look at the expression again, substituting for the prices as given:

$$p_1^0(\theta - 1)(q_1^1 - q_1^0) + p_2^0(\theta - 1)(q_2^1 - q_2^0) + \sum_{i=3}^{n} \underbrace{(p_i^1 - p_i^0)}_{=0}(q_i^1 - q_i^0) \geq 0$$

And so now, rearranging (and remember that theta is greater than 1, so dividing won't change the direction of the inequality):

$$(\theta - 1)\left[p_1^0(q_1^1 - q_1^0) + p_2^0(q_2^1 - q_2^0)\right] \geq 0 \Rightarrow \left[p_1^0(q_1^1 - q_1^0) + p_2^0(q_2^1 - q_2^0)\right] \geq 0$$

4. **Production and Cost**

All the firms in an industry have the same CRS production function. Let $z(r,q)$ be the cost minimizing input vector when output is q. Firms in the industry are all price takers.

(a) Show that $E(z_j,q)=1$.

(b) Show that $\frac{\partial}{\partial r_i}MC=\frac{\partial z_i}{\partial q}$.

(c) If the j-th input price were to rise, would it necessarily be the case that the equilibrium price would rise and output would fall?

(d) Let $A\hat{C}$ be the average cost after an increase in the price of input j from r_j to $\hat{r}_j$.

Consider the arc elasticity $\frac{r_j}{AC}\frac{A\hat{C}-AC}{\hat{r}_j-r_j}$. Show that this cannot exceed the input share $\frac{r_j z_j}{C}$.

(e) What conclusion can you draw about the (arc) equilibrium output price elasticity $\frac{r_j}{p}\frac{(\hat{p}-p)}{(\hat{r}_j-r_j)}$?

What is the point elasticity $E(p,r_j)$?

(no official solution delivered)