CLASSICAL MODEL

OLS: Ass: $E(Y|X)=X\beta \quad V(Y|X)=\sigma^2 I_n \quad \Pr[rk(X'X)=k]=1$

$$\widehat{\beta}_{OLS}=(X'X)^{-1}X'Y \quad \hat{Y}=PY \quad \hat{e}=MY \quad X'\hat{e}=0 \quad \text{nx } \widehat{\beta}_1=(X_1'M_2X_1)^{-1}X_1'M_2Y$$

$$r=\frac{\sum(X_i-\overline{X})(Y_i-\overline{Y})}{\sqrt{\sum(X_i-\overline{X})^2}\sqrt{\sum(Y_i-\overline{Y})^2}},\ \text{1x } \widehat{\beta}_1=r_{xy}\frac{s_y}{s_x},\ R^2=r^2,\ \text{2x } \widehat{\beta}_2=\frac{\widehat{\beta}_{12}-\widehat{\beta}_{13}\widehat{\beta}_{23}}{1-\widehat{\beta}_{23}\widehat{\beta}_{32}},\ \widehat{\beta}_{ij}=r_{ij}\frac{s_i}{s_j}$$

$$R^2_{UC}=1-\frac{\widehat{e}'\widehat{e}}{Y'Y}=\frac{\widehat{Y}'\widehat{Y}}{Y'Y}\in[0,1] \qquad R^2=R^2_C=1-\frac{\widehat{e}'\widehat{e}}{Y'M_0Y}\in[0,1] \text{ if c } R^2_{adj}=1-(1-R^2)\frac{n-1}{n-k}$$

Properties: $\Gamma\widehat{\beta}-BLUE: E\left(\widehat{\beta}|X\right)=\beta, Var\left(\widehat{\beta}|X\right)=\sigma^2(X'X)^{-1}, Cov\left(\widehat{\beta},\widehat{e}|X\right)=0$

$$Var\left(\widehat{\beta}_1|X\right)=\sigma^2(X_1'M_2X_1)^{-1}, \widehat{\sigma}^2_{OLS}=\frac{\widehat{e}'\widehat{e}}{n-k}\text{–unbiased}, V\left(\widehat{Y}_0|X\right)=\sigma^2\left(I+X_0(X'X)^{-1}X_0'\right)$$

Ass: Joint Normality: $Y|X\sim N_n(X\beta,\sigma^2I_n) \implies$ RandSamp: $(X,Y)_i\sim i.i.d.$

$$\Gamma\widehat{\beta}|X\sim N_k\left(\Gamma\beta,\sigma^2\Gamma(X'X)^{-1}\Gamma'\right), W=\left(\Gamma\widehat{\beta}-\Gamma\beta\right)'\left(\sigma^2\Gamma(X'X)^{-1}\Gamma'\right)^{-1}\left(\Gamma\widehat{\beta}-\Gamma\beta\right)\sim\chi^2(rk(\Gamma))$$

$$\frac{\Gamma\widehat{\beta}-\Gamma\beta}{\sqrt{\widehat{\sigma}^2\Gamma(X'X)^{-1}\Gamma'}}\sim t(rk(\Gamma)), \frac{1}{p}\left(\Gamma\widehat{\beta}-\Gamma\beta\right)'\left(\widehat{\sigma}^2\Gamma(X'X)^{-1}\Gamma'\right)^{-1}\left(\Gamma\widehat{\beta}-\Gamma\beta\right)\sim F(rk(\Gamma),n-k)$$

Hyp SignLevel=Pr (Rejected|null) = Pr (I error), Power=Pr (Rejected|alter) = 1 − Pr (II error)

$$R:\Gamma\widehat{\beta}=\gamma, F_0=\frac{RSS_R-RSS_U}{RSS_U}\frac{n-k}{p}=\frac{R^2_U-R^2_R}{1-R^2_U}\frac{n-k}{p}, \widehat{\beta}_R=\widehat{\beta}_U-(X'X)^{-1}\Gamma'\left(\Gamma(X'X)^{-1}\Gamma'\right)^{-1}\left(\Gamma\widehat{\beta}-\Gamma\beta\right)$$

MLE $\widehat{\beta}_{ML}=\widehat{\beta}_{OLS}=(X'X)^{-1}X'Y$, *attains* CLRB, $\widehat{\sigma}^2_{ML}=\frac{\widehat{e}'\widehat{e}}{n}$ - biased, does not

$$I(\beta,\sigma^2|X)^{-1}=-E\left[\frac{\partial^2L}{\partial\theta\partial\theta'}|X\right]^{-1}=\begin{bmatrix}\sigma^2(X'X)^{-1} & 0\\ 0 & 2\sigma^4/n\end{bmatrix}, \quad \sqrt{n}\left(\widehat{\theta}_{ML}-\theta\right)\xrightarrow{d}N\left(0,\lim\frac{1}{n}I_n^{-1}\right)$$

Tst: $\frac{W}{n}=h(\theta)'\left\{\frac{\partial Q'}{\partial\theta}\left[\frac{\partial^2Q}{\partial\theta\partial\theta'}\right]^{-1}\frac{\partial Q}{\partial\theta}\right\}^{-1}h(\theta)\Bigg|_{\widehat{\theta}_U}, \frac{LM}{n}=\frac{\partial Q'}{\partial\theta}\left[\frac{\partial^2Q}{\partial\theta\partial\theta'}\right]^{-1}\frac{\partial Q}{\partial\theta}\Bigg|_{\widehat{\theta}_R}, \frac{LR}{n}=2\left(Q\left(\widehat{\theta}_R\right)-Q\left(\widehat{\theta}_U\right)\right)$

CNLR: $W=n\frac{RSS_R-RSS_U}{RSS_U}, LM=n\frac{RSS_R-RSS_U}{RSS_R}, LR=n\ln\frac{RSS_R}{RSS_U}, LM\le LR\le W, all\sim\chi^2(rk(\Gamma))$

Deviations Multicol: $Var\left(\widehat{\beta}_j|X\right)=\sigma^2\left(X_j'M_{-j}X_j\right)^{-1}=\frac{\sigma^2}{\left(1-R_j^2\right)(X'M_jX)}$

Redund: no bias Omit:bias:$(X_1'X_1)^{-1}X_1X_2\beta_2$, Both:Vtrue↑,Vest?

Heter:$V(X), Var\left(\widehat{\beta}|X\right)=\sigma^2(X'X)^{-1}X'V(X)X(X'X)^{-1}, \widehat{\beta}_{GLS}=(X'VX)^{-1}X'VY$ - BLUE

AR(1): $\varepsilon_i=\phi\varepsilon_{i-1}+u_i \qquad Y_i-\phi Y_{i-1}=(X_i-\phi X_{i-1})\beta+u_i \qquad Y_1=\sqrt{1-\phi^2}X_1\beta+u_1$

ASYMPTOTICS

Asy prop OLS Ass.$Y_i=X_i\beta+\varepsilon_i$, $\{X_i,\varepsilon_i\}$ *i.i.d.*, $E[X_i\varepsilon_i]=0$, $E|X_i\varepsilon_i|<\infty$, $E|X_i|^2<\infty$ $E[X_i'X_i]$ *p.s.d.*

Consistent:$\widehat{\beta}=\beta+\left(\frac{1}{n}\sum X_i'X_i\right)^{-1}\left(\frac{1}{n}\sum X_i'\varepsilon_i\right)\xrightarrow{p}\beta$, Ass: $same+E|X_i\varepsilon_i|^2<\infty, E[X_i'X_i\varepsilon_i^2]\ p.s.d.$

$$\sqrt{n}\left(\widehat{\beta}-\beta\right)=\sqrt{n}\left(\frac{1}{n}\sum X_i'X_i\right)^{-1}\left(\frac{1}{n}\sum X_i'\varepsilon_i\right)\xrightarrow{d}N\left(0,E[X_i'X_i]^{-1}E[X_i'X_i\varepsilon_i^2]E[X_i'X_i]^{-1}\right)$$

i.i.d.: $\widehat{\beta}\overset{A}{\sim}N\left(0,\frac{1}{n}AV\left(\widehat{\beta}\right)\right), \frac{1}{n}\widehat{AV\left(\widehat{\beta}\right)}=\frac{1}{n}\left[\frac{1}{n}\sum X_i'X_i\right]^{-1}\left[\frac{1}{n}\sum X_i'X_i\varepsilon_i^2\right]\left[\frac{1}{n}\sum X_i'X_i\right]^{-1}$ – **White**

$$t_0=\frac{\Gamma\widehat{\beta}-\Gamma\beta_0}{\sqrt{\Gamma\frac{1}{n}\widehat{AV\left(\widehat{\beta}\right)}\Gamma'}}\xrightarrow{d}N(0,1) \qquad F_0=\left(\Gamma\widehat{\beta}-\Gamma\beta_0\right)'\left(\Gamma\frac{1}{n}\widehat{AV\left(\widehat{\beta}\right)}\Gamma'\right)^{-1}\left(\Gamma\widehat{\beta}-\Gamma\beta_0\right)\xrightarrow{d}\chi^2(rk(\Gamma))$$

$W_0=\left(\gamma\left(\widehat{\beta}\right)-\gamma(\beta)\right)'\left(\Gamma\left(\widehat{\beta}\right)\frac{1}{n}\widehat{AV\left(\widehat{\beta}\right)}\Gamma\left(\widehat{\beta}\right)'\right)^{-1}\left(\gamma\left(\widehat{\beta}\right)-\gamma(\beta)\right)\xrightarrow{d}\chi^2(rk(\Gamma))$ – **Wald**

Asy WLS:$Y_i=X_i\beta+\varepsilon_i$, $\{X_i,\varepsilon_i\}$ *i.i.d.*, $E[X_i|\varepsilon_i]=0$, *etc* $\varepsilon_i^2=Z_i\alpha+u_i$, $\widehat{\varepsilon_i^2}$,

$\widehat{\beta}_{FWLS}=\beta+\left(\frac{1}{n}\sum\frac{X_i'X_i}{\widehat{\varepsilon_i^2}}\right)^{-1}\left(\frac{1}{n}\sum\frac{X_i'\varepsilon_i}{\widehat{\varepsilon_i^2}}\right)\xrightarrow{p}\beta_{WLS}\xrightarrow{p}\beta$ asy more efficient then OLS

EXTREMUM ESTIMATORS

Extr.Est: $\widehat{\theta} = \arg\max_{\theta\in\Theta} Q_n(\theta)$ $\quad$ $Lem: \underline{\Theta compact, Q_n continuous\ measurable}_1 \Longrightarrow \exists\widehat{\theta}$

$Th1: \Theta Q_n same_1, \underline{\exists Q_0(\theta): \exists!\theta_0 = \arg\max_{\theta\in\Theta} Q_0(\theta), Q_n \overset{p}{\rightrightarrows} Q_0}_2 \Longrightarrow \widehat{\theta} \overset{p}{\to} \theta_0$

$Th2: \underline{\theta_0 \in \Theta interior}, Q_n \underline{concave\ } meas, \exists Q_0(\theta): \exists!\theta_0 = \arg\max Q_0(\theta), \underline{Q_n \overset{p}{\to} Q_0} \Longrightarrow \widehat{\theta} \overset{p}{\to} \theta_0$

M-type: $Q_n(\theta) = \frac{1}{n}\sum m(W_i,\theta)$

$Th3: X_i i.i.d., \Theta compact, m(W,\theta) \in C^0, E[\sup_{\theta\in\Theta}|m(W_i,\theta)|] < \infty \overset{ULLN}{\Longrightarrow} \widehat{\theta} \overset{p}{\to} \theta_0$

$Th4: i.i.d., \theta_0 \in \Theta int, m(W,\theta)\, concave, E[|m(W_i,\theta)|] < \infty \overset{LLN}{\Longrightarrow} \widehat{\theta} \overset{p}{\to} \theta_0$

$Th5: i.i.d., \theta_0 \in \Theta int, m(W,\theta) \in C^2, \widehat{\theta} \overset{p}{\to} \theta_0, E\left[\sup\left|\frac{\partial^2 m(W_i,\theta)}{\partial\theta\partial\theta'}\right|\right] < \infty, 0 < E\left[\frac{\partial^2 m(W_i,\theta)}{\partial\theta\partial\theta'}\right] < \infty,$

$\sum \frac{\partial m(W_i,\theta)}{\sqrt{n}\partial\theta} \overset{d}{\to} N\left(0, E\left[\frac{\partial m(W_i,\theta)}{\partial\theta}\frac{\partial m(W_i,\theta)}{\partial\theta'}\right] p.s.d.\right) \overset{CLT}{\Longrightarrow} \sqrt{n}\left(\widehat{\theta}-\theta_0\right) \overset{d}{\to} N\left(0, H_0^{-1}\Omega_0 H_0^{-1}\right)$

NONLINEAR LEAST SQUARES

NLS: Ass: $\underline{Y_i = g(X_i,\beta_0)+\varepsilon_i,\ \{X_i,\varepsilon_i\}\, i.i.d.,\ E[\varepsilon_i|X_i]=0,\ \beta\neq\beta_0 \Longrightarrow g(X_i,\beta)\neq g(X_i,\beta_0)}$

$m(W_i,\beta) = (Y_i - g(X_i,\beta))^2$, Th1. $Bcomp, g(X,\beta)\in C^0, E[\sup||m(W_i,\beta)||] < \infty \Longrightarrow \widehat{\beta} \overset{p}{\to} \beta_0$

$s(W_i,\beta) = -\frac{\partial g(W_i,\beta)}{\partial\beta}(Y_i - g(X_i,\beta)),\ H(W_i,\beta) = \frac{\partial g(W_i,\beta)}{\partial\beta}\frac{\partial g(W_i,\beta)}{\partial\beta} - \frac{\partial^2 g(W_i,\beta)}{\partial\beta\partial\beta'}(Y_i - g(X_i,\beta))$

Th2: $i.i.d., \beta_0 \in Bint, m(W,\beta)\in C^2, E[\sup|H(W_i,\beta)|] < \infty, 0 < E\left[s(W_i,\beta_0)^2\right] < \infty,$

$0 < H_0 = E[H(W_i,\beta_0)] < \infty, \widehat{\beta} \overset{p}{\to} \beta_0 \quad \overset{CLT}{\Longrightarrow} \sqrt{n}\left(\widehat{\beta}-\beta_0\right) \overset{d}{\to} N\left(0, H_0^{-1}\Omega_0 H_0^{-1}\right)$

MAXIMUM LIKELIHOOD

ML: Ass: $\underline{f(w_i,\theta_0),\ w_i i.i.d., m(W_i,\beta) = \ln f(w_i,\theta)}$, K-L: $\theta_0 id \Longrightarrow E[\ln f(w_i,\theta)] < E[\ln f(w_i,\theta_0)]$

$Th1: i.i.d., \Theta compact, \theta_0 id, \ln f(w_i,\theta)\in C^0, E[\sup_{\theta\in\Theta}||\ln f(w_i,\theta)||] < \infty \Longrightarrow \widehat{\theta}_{ML} \overset{p}{\to} \theta_0$

$Th2: i.i.d., \theta_0 \in \Theta int, \ln f(w_i,\theta) > 0 \in C^2, \int\sup\left|\frac{\partial f(w_i,\theta)}{\partial\theta}\right| dw_i < \infty, \int\sup\left|\frac{\partial^2 f(w_i,\theta)}{\partial\theta\partial\theta'}\right| dw_i < \infty,$

$E\left[\sup\left|\frac{\partial^2 \ln f(w_i,\theta)}{\partial\theta\partial\theta'}\right|\right] < \infty, 0 < i(\theta_0) = -E\left[\left(\frac{\partial^2 \ln f(w_i,\theta)}{\partial\theta\partial\theta'}\right)^2\right] < \infty, \widehat{\theta} \overset{p}{\to} \theta_0$

$$\overset{CLT}{\Longrightarrow} \sqrt{n}\left(\widehat{\theta}_{ML}-\theta_0\right) \overset{d}{\to} N\left(0, i(\theta_0)^{-1}\right)$$

TESTS $\frac{W}{n} = h(\theta)'\left\{\frac{\partial h}{\partial\theta}'\widehat{V}^{-1}\frac{\partial h}{\partial\theta}\right\}^{-1} h(\theta)\Big|_{\widehat{\theta}_U},\ \frac{LM}{n} = \frac{\partial Q_n}{\partial\theta}'\left[\frac{\partial^2 Q_n}{\partial\theta\partial\theta'}\right]^{-1}\frac{\partial Q_n}{\partial\theta}\Big|_{\widehat{\theta}_R},\ \frac{LR}{n} = 2\left(Q_n\left(\widehat{\theta}_R\right) - Q_n\left(\widehat{\theta}_U\right)\right)$

all $\sim \chi^2(rk(\Gamma))$ $\quad$ ex: $Q_n = \frac{1}{n}\sum\ln f(w_i,\theta)$

SAMPLE SELECTION MODELS

Qualit.Resp $\Pr(y_i = 1|x_i) =:$ LPM: $x_i\beta_0$, Probit: $\int_{-\infty}^{x_i\beta_0}\frac{-e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}du$, Logit: $\frac{e^{x_i\beta_0}}{1+e^{x_i\beta_0}}$, LogWeib: $e^{-e^{x_i\beta_0}}$

$E[y_i|x_i] = F(x_i\beta_0),\ NLS: \sum(y_i - F(x_i\beta))^2,\ WNLS: \sum\frac{(y_i - F(x_i\beta))^2}{F(x_i\beta)(1-F(x_i\beta))},$

$\underline{ML: \prod F(x_i\beta)^{y_i}(1-F(x_i\beta))^{1-y_i}}$, MargEff: $f(\overline{x}\beta)\beta$, δ-met. Compare: Logit*0.5513

Censored $y_i = \max[x_i\beta_0 + \varepsilon_i, 0]$: $\varepsilon_i|x_i \sim N(0,\sigma_0^2)$ $\quad$ $E[y_i|x_i] = \left[x_i\beta_0 + \sigma_0\frac{\phi(x_i\beta_0)}{\Phi(x_i\beta_0)}\right]\Phi(x_i\beta_0)$

$Var[y_i|x_i] = \sigma_0^2\left[1 + \frac{x_i\beta_0\phi\left(\frac{x_i\beta_0}{\sigma_0}\right)}{\sigma_0\Phi\left(\frac{x_i\beta_0}{\sigma_0}\right)} - \left(\frac{\phi\left(\frac{x_i\beta_0}{\sigma_0}\right)}{\Phi\left(\frac{x_i\beta_0}{\sigma_0}\right)}\right)^2\right]\Phi\left(\frac{x_i\beta_0}{\sigma_0}\right)^2$

$NLS, WNLS, ML: \prod\left[1-\Phi\left(\frac{x_i\beta}{\sigma}\right)\right]^{1-d_i}\left(\frac{1}{\sigma}\phi\left(\frac{y_i - x_i\beta}{\sigma}\right)\right)^{d_i}, Heck2S: probit\widehat{\gamma} \to x + \widehat{\lambda}(-x\widehat{\gamma})$

MargEff:expected $(1+\lambda'(-x\gamma_0))\beta$, realiz: $\beta\Phi(x\gamma_0)$

Truncated $y_i = x_i\beta_0 + u_i, \varepsilon_i|x_i \sim N(0,\sigma_0^2)$ $\quad$ $f[y_i|x_i] = \frac{\phi\left(\frac{x_i\beta_0}{\sigma_0}\right)}{\sigma_0\Phi\left(\frac{x_i\beta_0}{\sigma_0}\right)} ML: \prod\frac{\frac{1}{\sigma}\phi\left(\frac{y_i - x_i\beta}{\sigma}\right)}{\Phi\left(\frac{x_i\beta}{\sigma}\right)}$