Econ203B MT03 Problem 2 Anton Cheremukhin

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$y_i = x_i^* \beta + \varepsilon_i \qquad x_i = x_i^* \nu_i \qquad z_i = x_i^* \eta_i \qquad x_i^*, \varepsilon_i, \nu_i, \eta_i$ independent, each i.i.d.

(a) $\beta_{OLS} = \beta + \frac{\Sigma x_i y_i - \beta x_i^2}{\Sigma x_i^2} = \beta + \frac{\Sigma\left[x_i^*\nu_i\left(x_i^*\beta+\varepsilon_i\right)-\beta\left(x_i^*\nu_i\right)^2\right]}{\Sigma\left(x_i^*\nu_i\right)^2} = \beta + \frac{\Sigma\left[\left(x_i^*\right)^2\left(\left(\nu_i-(\nu_i)^2\right)\beta+\nu_i\varepsilon_i\right)\right]}{\Sigma\left(x_i^*\right)^2(\nu_i)^2}$

If $0 < E\left|x_i^*\right|^2 < \infty, \qquad E\left|\varepsilon_i\right|^2 < \infty, \qquad E\varepsilon_i = 0$ and $0 < E\nu_i^2 = E\nu_i < \infty$

then $\beta_{OLS} \xrightarrow{p} \beta$.

(b) $\tilde{\beta} = \beta + \frac{\Sigma(\alpha x_i + (1-\alpha)z_i)y_i - \beta x_i z_i}{\Sigma x_i z_i} = \beta + \frac{\Sigma\left[(\alpha\nu_i+(1-\alpha)\eta_i-\nu_i\eta_i)x_i^*\left(x_i^*\beta+\varepsilon_i\right)+x_i^*\varepsilon_i\nu_i\eta_i\right]}{\Sigma\left(x_i^*\right)^2\nu_i\eta_i}$

$E\left[\alpha\nu_i + (1-\alpha)\eta_i - \nu_i\eta_i\right] = \alpha E\nu_i + (1-\alpha)E\eta_i - E\nu_i E\eta_i = 0$

If $0 < E\left|x_i^*\right|^2 < \infty, \qquad E\left|\varepsilon_i\right|^2 < \infty, \qquad 0 < E\left|\nu_i\right|^2 < \infty, \qquad 0 < E\left|\eta_i\right|^2 < \infty$ and $E\varepsilon_i = 0$

then $\beta_{OLS} \xrightarrow{p} \beta$.

(c) $\tilde{\beta} = \beta + \frac{\Sigma(\alpha x_i + (1-\alpha)z_i)y_i - \beta x_i z_i}{\Sigma x_i z_i} = \beta + \frac{\Sigma\left[(\alpha\nu_i+(1-\alpha)\eta_i-\nu_i\eta_i)x_i^*\left(x_i^*\beta+\varepsilon_i\right)+x_i^*\varepsilon_i\nu_i\eta_i\right]}{\Sigma\left(x_i^*\right)^2\nu_i\eta_i}$

Denote $\qquad A_i = \left(\alpha\nu_i + (1-\alpha)\eta_i - \nu_i\eta_i\right) \qquad B_i = x_i^*\left(x_i^*\beta + \varepsilon_i\right) \qquad C_i = x_i^*\varepsilon_i\nu_i\eta_i$

.

$D = E\left[\left(x_i^*\right)^2\nu_i\eta_i\right] = E\left[\left(x_i^*\right)^2\right]$

.

$G = Var\left[A_iB_i + C_i\right] = Var\left[A_iB_i\right] + Var\left[C_i\right] + 2Cov\left[A_iB_i, C_i\right]$

.

$Cov\left[A_iB_i, C_i\right] = E\left[\left(A_iB_i - E\left[A_iB_i\right]\right)\left(C_i - E\left[C_i\right]\right)\right] = E\left[A_iB_iC_i\right] - E\left[A_iB_i\right]E\left[C_i\right] =$

$= E\left[A_iB_iC_i\right] = E\left[\left(\alpha\nu_i\eta_i\nu_i + (1-\alpha)\nu_i\eta_i\eta_i - \nu_i\eta_i\nu_i\eta_i\right)\left(x_i^*\varepsilon_i\right)^2\right] =$

$= \left(\alpha E\left[\nu_i^2\right] + (1-\alpha)E\left[\eta_i^2\right] - E\left[\nu_i^2\right]E\left[\eta_i^2\right]\right)E\left[\left(x_i^*\right)^2\right]E\left[\varepsilon_i^2\right]$

.

$Var\left[A_iB_i\right] = E\left[\left(A_iB_i\right)^2\right] - \left(E\left[A_iB_i\right]\right)^2 = E\left[B_i^2\right]E\left[A_i^2\right] - \left(E\left[A_i\right]\right)^2\left(E\left[B_i\right]\right)^2 = E\left[B_i^2\right]E\left[A_i^2\right]$

.

$Var\left[C_i\right] = E\left[C_i^2\right] = E\left[\nu_i^2\right]E\left[\eta_i^2\right]E\left[\left(x_i^*\right)^2\right]E\left[\varepsilon_i^2\right]$

.

$F = E\left[B_i^2\right] = \beta^2 E\left[\left(x_i^*\right)^4\right] + E\left[\left(x_i^*\right)^2\right]E\left[\varepsilon_i^2\right]$

.

$E\left[A_i^2\right] = \alpha^2 E\left[\nu_i^2\right] + (1-\alpha)^2 E\left[\eta_i^2\right] + E\left[\nu_i^2\right]E\left[\eta_i^2\right] + 2\alpha(1-\alpha) - 2\alpha E\left[\nu_i^2\right] - 2(1-\alpha)E\left[\eta_i^2\right]$

.

$G = \left\{\alpha^2 E\left[\nu_i^2\right] + (1-\alpha)^2 E\left[\eta_i^2\right] + E\left[\nu_i^2\right]E\left[\eta_i^2\right] + 2\alpha(1-\alpha) - 2\alpha E\left[\nu_i^2\right] - 2(1-\alpha)E\left[\eta_i^2\right]\right\} *$
$\left\{\beta^2 E\left[\left(x_i^*\right)^4\right] + E\left[\left(x_i^*\right)^2\right]E\left[\varepsilon_i^2\right]\right\} + 2\left\{\alpha E\left[\nu_i^2\right] + (1-\alpha)E\left[\eta_i^2\right]\right\} * E\left[\left(x_i^*\right)^2\right]E\left[\varepsilon_i^2\right] -$
$-E\left[\nu_i^2\right]E\left[\eta_i^2\right]E\left[\left(x_i^*\right)^2\right]E\left[\varepsilon_i^2\right]$

.

$$\sqrt{n}\left(\tilde{\beta} - \beta\right) \xrightarrow{d} N\left(0, D^{-1}GD^{-1}\right)$$

(c) $\qquad \min_\alpha G(\alpha)$

$\left\{\alpha E\left[\nu_i^2\right] - (1-\alpha)E\left[\eta_i^2\right] + (1-2\alpha) - E\left[\nu_i^2\right] + E\left[\eta_i^2\right]\right\}F + \left(E\left[\nu_i^2\right] - E\left[\eta_i^2\right]\right)E\left[\left(x_i^*\right)^2\right]E\left[\varepsilon_i^2\right] = 0$

$$\alpha^* = \frac{1}{E\left[\nu_i^2\right]-1+E\left[\eta_i^2\right]-1}\left[E\left[\nu_i^2\right] - 1 + \frac{E\left[\eta_i^2\right]-E\left[\nu_i^2\right]}{\frac{E\left[\left(x_i^*\right)^4\right]}{E\left[\left(x_i^*\right)^2\right]E\left[\varepsilon_i^2\right]}\beta^2+1}\right]$$