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Exercise 1

$$P' = (X(X'X)^{-1}X')' = (X')'((X'X)^{-1})'X' = X(X'(X')')^{-1}X' = P$$
$$PP = X(X'X)^{-1}X'X(X'X)^{-1}X' = X(X'X)^{-1}X' = P$$
$$M' = (I-P)' = I - P' = I - P = M \qquad MM = (I-P)(I-P) = I - 2P + PP = I - P = M$$
$$PX = PX = X(X'X)^{-1}X'X) = X, \;\; MX = (I-P)X = (X - X(X'X)^{-1}X'X) = X - X = 0$$

Exercise 2

$$\varepsilon \sim N(0, \sigma^2 I), \qquad \widehat{\beta} = (X'X)^{-1}X'y = (X'X)^{-1}X'(X\beta+\varepsilon) = \beta + (X'X)^{-1}X'\varepsilon$$
$$\widehat{\beta} \sim N\left(\beta, \sigma^2(X'X)^{-1}X'X(X'X)^{-1}\right) = N\left(\beta, \sigma^2(X'X)^{-1}\right)$$
$$e = M\varepsilon \sim N(0, \sigma^2 MM') = N(0, \sigma^2 MM) = N(0, \sigma^2 M)$$

Hence, e and $\widehat{\beta}$ have joint normal distribution.

$$E[(X'X)^{-1}X'\varepsilon \cdot (M\varepsilon)'] = \sigma^2(X'X)^{-1}X'M' = \sigma^2(X'X)^{-1}(MX)' = 0$$

Therefore, they are independent.

$$\text{trace}(M) = \text{trace}(I-P) = \text{trace}(I) - \text{trace}(P) = n-k$$
$$\frac{e'e}{\sigma^2} = \frac{\varepsilon' M\varepsilon}{\sigma^2} \sim \chi^2(n-k), \qquad E\left[\frac{e'e}{\sigma^2}\right] = n-k \qquad \Rightarrow \qquad E\left[\frac{e'e}{n-k}\right] = \sigma^2.$$

Exercise 3

Because Φ is monotonically increasing, we have $Y \leq y$ if and only if $\Phi(Y) \leq \Phi(y)$. Hence, $\Pr[Y \leq y] = \Pr[\Phi(Y) \leq \Phi(y)] = \Pr[\Phi(\Phi^{-1}(U)) \leq \Phi(y)] = \Pr[U \leq \Phi(y)] = \Phi(y)$. Therefore, if $U \sim U(0,1)$, then $\Phi^{-1}(U)$ has c.d.f. Φ, i.e. $\Phi^{-1}(U) \sim N(0,1)$.

Exercise 4

$X_n \sim \chi^2(n)$. If we had $Y_i \overset{iid}{\sim} N(0,1)$, then $\sum_{i=1}^n Y_i^2 \sim \chi^2(n)$.

We can treat X_n as a sum of some i.i.d. normal Y_i^2 because their distributions are equivalent.

By construction $Y_i \overset{iid}{\sim} N(0,1)$ and $Z_i = Y_i^2 \overset{iid}{\sim} \chi^2(1)$. $\qquad E[Z_i] = 1 \qquad Var[Z_i] = 2$

Replacing $\frac{X_n - n}{\sqrt{n}} = \frac{\sum_{i=1}^n Y_i^2 - n}{\sqrt{n}} = \sqrt{n}(\overline{Y^2} - 1) = \sqrt{n}(\overline{Z} - 1) = \sqrt{n}(\overline{Z} - E[Z])$

By LLN $\sqrt{n}(\overline{Z} - E[Z]) \xrightarrow{d} N(0, Var[Z]) = N(0,2)$.

Exercise 5

$$Z = \begin{bmatrix} Y \\ X \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_Y \\ \mu_X \end{bmatrix}, \begin{bmatrix} \sigma_Y^2 & \rho\sigma_Y\sigma_X \\ \rho\sigma_Y\sigma_X & \sigma_X^2 \end{bmatrix}\right) = N(\mu, \Sigma)$$
$$\widetilde{Z} = Z - \mu = \begin{bmatrix} Y - \mu_Y \\ X - \mu_X \end{bmatrix} \sim N(0, \Sigma)$$
$$\begin{bmatrix} U \\ X - \mu_X \end{bmatrix} = \begin{bmatrix} (Y-\mu_Y) - \rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X) \\ X - \mu_X \end{bmatrix} = \begin{bmatrix} 1 & -\rho\frac{\sigma_Y}{\sigma_X} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y - \mu_Y \\ X - \mu_X \end{bmatrix} = A\widetilde{Z}$$
$$A\widetilde{Z} \sim N(A0, A\Sigma A') = N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_Y^2 - \rho^2\sigma_Y^2 & 0 \\ 0 & \sigma_X^2 \end{bmatrix}\right) \qquad \Rightarrow \qquad VarU = \sigma_Y^2 - \rho^2\sigma_Y^2$$

U and $X-\mu_X$ have joint normal distribution with zero covariance and hence are independent.
$Y=\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(X-\mu_X)+U$
$Y|(X=x)=\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)+U\sim N\left(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X),VarU\right)$
$E\left[Y^2|X=x\right]=Var\left[Y|X=x\right]+E\left[Y|X=x\right]^2=\sigma_Y^2-\rho^2\sigma_Y^2+\left(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right)^2$

Exercise 6

$$X_i\overset{iid}{\sim}N(\mu,\sigma^2)\qquad\Rightarrow\qquad X_i-\mu\overset{iid}{\sim}N(0,\sigma^2)\qquad\Rightarrow\qquad Z_i=\frac{X_i-\mu}{\sigma}\overset{iid}{\sim}N(0,1)$$

$$X_i=\mu+\sigma Z_i\qquad\overline{X}=\frac{1}{n}\sum_{i=1}^n X_i=\mu+\sigma\frac{1}{n}\sum_{i=1}^n Z_i=\mu+\sigma\overline{Z}$$

$$Y_n=\frac{\frac{1}{n}\sum_{i=1}^n\left(X_i-\overline{X}\right)^3}{\left(\frac{1}{n}\sum_{i=1}^n\left(X_i-\overline{X}\right)^2\right)^{3/2}}=\frac{\frac{1}{n}\sum_{i=1}^n\left(\mu+\sigma Z_i-\mu-\sigma\overline{Z}\right)^3}{\left(\frac{1}{n}\sum_{i=1}^n\left(\mu+\sigma Z_i-\mu-\sigma\overline{Z}\right)^2\right)^{3/2}}=\frac{\frac{1}{n}\sum_{i=1}^n\left(Z_i-\overline{Z}\right)^3}{\left(\frac{1}{n}\sum_{i=1}^n\left(Z_i-\overline{Z}\right)^2\right)^{3/2}}$$

$$\underset{n\to\infty}{plim}\left(\frac{1}{n}\sum_{i=1}^n\left(Z_i-\overline{Z}\right)^2\right)^{3/2}=\underset{n\to\infty}{plim}\left(\frac{1}{n}\sum_{i=1}^n Z_i{}^2-\left(\frac{1}{n}\sum_{i=1}^n Z_i\right)^2\right)^{3/2}=$$

$$\left(\underset{n\to\infty}{plim}\frac{1}{n}\sum_{i=1}^n Z_i{}^2-\left(\underset{n\to\infty}{plim}\frac{1}{n}\sum_{i=1}^n Z_i\right)^2\right)^{3/2}=\left(E\left[Z_i{}^2\right]-\left(E\left[Z_i\right]\right)^2\right)^{3/2}=1$$

$$E\left[Z^{2k}\right]=\frac{(2k)!}{2^k k!}\qquad E\left[Z^{2k-1}\right]=0\qquad\left.\frac{1}{2^k}\frac{(2k)!}{(k)!}\right|_{k=2}=3\qquad\left.\frac{1}{2^k}\frac{(2k)!}{(k)!}\right|_{k=3}=15$$

$$E\begin{bmatrix}Z_i^3\\Z_i^2\\Z_i\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}\qquad Var\begin{bmatrix}Z_i^3\\Z_i^2\\Z_i\end{bmatrix}=E\begin{bmatrix}Z_i^6&Z_i^5&Z_i^4\\Z_i^5&Z_i^4&Z_i^3\\Z_i^4&Z_i^3&Z_i^2\end{bmatrix}-\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0&1&0\end{bmatrix}=$$

$$\begin{bmatrix}15&0&3\\0&3&0\\3&0&1\end{bmatrix}-\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}=\begin{bmatrix}15&0&3\\0&2&0\\3&0&1\end{bmatrix}$$

The vector is i.i.d. with finite expectation and variance. Hence, by LLN

$$\sqrt{n}\left(\begin{bmatrix}\frac{1}{n}\sum_{i=1}^n Z_i^3\\\frac{1}{n}\sum_{i=1}^n Z_i^2\\\frac{1}{n}\sum_{i=1}^n Z_i\end{bmatrix}-\begin{bmatrix}0\\1\\0\end{bmatrix}\right)\xrightarrow{d}N\left(\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}15&0&3\\0&2&0\\3&0&1\end{bmatrix}\right)$$

$$\begin{bmatrix}\frac{1}{n}\sum_{i=1}^n Z_i^3&\frac{1}{n}\sum_{i=1}^n Z_i^2&\frac{1}{n}\sum_{i=1}^n Z_i\end{bmatrix}'\equiv\begin{bmatrix}A_n&B_n&C_n\end{bmatrix}'$$

$$\frac{1}{n}\sum_{i=1}^n\left(Z_i-\overline{Z}\right)^3=A_n-3C_nB_n+2\left(C_n\right)^3=u\left(\begin{bmatrix}A_n&B_n&C_n\end{bmatrix}'\right)$$

$$G=\left.\frac{\partial u(x)}{\partial x}\right|_{(0,1,0)}=\left.\begin{bmatrix}1&-3C_n&-3B_n+6\left(C_n\right)^2\end{bmatrix}\right|_{(0,1,0)}=\begin{bmatrix}1&0&-3\end{bmatrix}$$

$$u\left(\begin{bmatrix}0&1&0\end{bmatrix}'\right)=0\qquad w=G\Sigma G'=\begin{bmatrix}1&0&-3\end{bmatrix}\begin{bmatrix}15&0&3\\0&2&0\\3&0&1\end{bmatrix}\begin{bmatrix}1\\0\\-3\end{bmatrix}=6$$

By delta-method: $\frac{1}{\sqrt{n}}\sum_{i=1}^n\left(Z_i-\overline{Z}\right)^3=\sqrt{n}\left(u\left(\begin{bmatrix}\frac{1}{n}\sum_{i=1}^n Z_i^3\\\frac{1}{n}\sum_{i=1}^n Z_i^2\\\frac{1}{n}\sum_{i=1}^n Z_i\end{bmatrix}\right)-0\right)\xrightarrow{d}N(0,6)$

Summing up, $\frac{1}{\sqrt{n}}\sum_{i=1}^n\left(Z_i-\overline{Z}\right)^3\xrightarrow{d}N(0,6)$, $\qquad\underset{n\to\infty}{plim}\left(\frac{1}{n}\sum_{i=1}^n\left(Z_i-\overline{Z}\right)^2\right)^{3/2}=1$

By continuity $\underset{n\to\infty}{plim}\frac{1}{\left(\frac{1}{n}\sum_{i=1}^n\left(Z_i-\overline{Z}\right)^2\right)^{3/2}=1}=1.$

By Slutsky, $\sqrt{n}Y_n=\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n\left(Z_i-\overline{Z}\right)^3}{\left(\frac{1}{n}\sum_{i=1}^n\left(Z_i-\overline{Z}\right)^2\right)^{3/2}}\xrightarrow{d}N(0,6)$.