ECONOMICS 203A
FINAL EXAMINATION
2004

Instruction: Answer every question. You are allowed to use a handheld calculator.

1. (10 pt.) Let X denote some $n \times k$ nonstochastic with full column rank. Let $P = X\left(X'X\right)^{-1}X'$, and $M = I_n - P$. Show that

$$P' = P, \quad PP = P, \quad M' = M,$$
$$MM = M, \quad PX = X, \quad MX = 0.$$

2. (10 pt.) Consider the linear regression model $y = X\beta + \epsilon$, where y, X, β, ϵ are $n \times 1$, $n \times k$, $k \times 1$, $n \times 1$ matrices, and $\epsilon \sim N\left(0, \sigma^2 I_n\right)$. We will assume that X is nonstochastic with full column rank. Let $\widehat{\beta} = \left(X'X\right)^{-1}X'y = \beta + \left(X'X\right)^{-1}X'\epsilon$, denote the OLS estimator, and let $e = y - X\widehat{\beta}$ denote the residual vector. Prove that

   (a) (5 pt.) $\widehat{\beta}$ and $e'e$ are independent of each other.

   (b) (5 pt.) $E\left[\frac{e'e}{n-k}\right] = \sigma^2$.

3. (10 pt.) Let U be a random variable such that its density is equal to 1 over $(0,1)$ and zero elsewhere. Let $\Phi(\cdot)$ denote the cumulative distribution function of $N(0,1)$. Prove that $\Phi^{-1}(U) \sim N(0,1)$.

4. (10 pt.) Suppose that $X_n \sim \chi^2(n)$. Prove that

$$\frac{X_n - n}{\sqrt{n}} \xrightarrow{d} N(0,2).$$

5. (10 pt.) Assume that

$$\begin{bmatrix} Y \\ X \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_Y \\ \mu_X \end{bmatrix}, \begin{bmatrix} \sigma_Y^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_X^2 \end{bmatrix}\right).$$

   (a) (5 pt.) Let

$$U = (Y - \mu_Y) - \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}(X - \mu_X) = (Y - \mu_Y) - \frac{\rho\sigma_Y}{\sigma_X}(X - \mu_X).$$

   Prove that U is independent of $X - \mu_X$. Write

$$Y = \mu_Y + \frac{\rho\sigma_Y}{\sigma_X}(X - \mu_X) + U,$$

   and conclude that the conditional distribution of Y given $X = x$ is equal to

$$N\left(\mu_Y + \frac{\rho\sigma_Y}{\sigma_X}(x - \mu_X), \mathrm{Var}(U)\right).$$

   What is $\mathrm{Var}(U)$?

   (b) (5 pt.) Compute

$$E\left[Y^2 \middle| X = x\right].$$

   You may use the fact that, if $Z \sim N(0,1)$, then $E[Z] = 0$, $E\left[Z^2\right] = 1$, and $E\left[Z^3\right] = 0$.

6. (10 pt.) Suppose that $X_1, \ldots, X_n$ are i.i.d. $N\left(\mu, \sigma^2\right)$. Let

$$Y_n = \frac{\frac{1}{n}\sum_{i=1}^n \left(X_i - \overline{X}\right)^3}{\left(\frac{1}{n}\sum_{i=1}^n \left(X_i - \overline{X}\right)^2\right)^{3/2}}.$$

You are asked to characterize the asymptotic distribution of $\sqrt{n}Y_n$. For this purpose, prove the following:

(a) (1 pt.) Let

$$Z_i = \frac{X_i - \mu}{\sigma}.$$

Show that $Z_i \overset{i.i.d.}{\sim} N(0,1)$, and

$$Y_n = \frac{\frac{1}{n}\sum_{i=1}^n \left(Z_i - \overline{Z}\right)^3}{\left(\frac{1}{n}\sum_{i=1}^n \left(Z_i - \overline{Z}\right)^2\right)^{3/2}}.$$

(b) (2 pt.) Show that

$$\operatorname*{plim}_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^n \left(Z_i - \overline{Z}\right)^2\right)^{3/2} = 1.$$

Hint: Write

$$\left(\frac{1}{n}\sum_{i=1}^n \left(Z_i - \overline{Z}\right)^2\right)^{3/2} = \left(\frac{1}{n}\sum_{i=1}^n Z_i^2 - \left(\frac{1}{n}\sum_{i=1}^n Z_i\right)^2\right)^{3/2}.$$

(c) (2 pt.) Show that

$$\sqrt{n}\left(\begin{bmatrix} \frac{1}{n}\sum_{i=1}^n Z_i^3 \\ \frac{1}{n}\sum_{i=1}^n Z_i^2 \\ \frac{1}{n}\sum_{i=1}^n Z_i \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \xrightarrow{d} N\left(0, \begin{bmatrix} 15 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}\right).$$

Hint: Because $Z_i \sim N(0,1)$, we have $E\left[Z_i^6\right] = 15$, $E\left[Z_i^5\right] = 0$, $E\left[Z_i^4\right] = 3$, $E\left[Z_i^3\right] = 0$, and $E\left[Z_i^2\right] = 1$.

(d) (4 pt.) Show that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n \left(Z_i - \overline{Z}\right)^3 \xrightarrow{d} N(0, \omega)$$

for some ω. What is ω? Hint: Write

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^n \left(Z_i - \overline{Z}\right)^3 &= \frac{1}{n}\sum_{i=1}^n Z_i^3 - 3\overline{Z}\cdot\left(\frac{1}{n}\sum_{i=1}^n Z_i^2\right) + 3\overline{Z}^2\cdot\left(\frac{1}{n}\sum_{i=1}^n Z_i\right) - \overline{Z}^3 \\
&= \frac{1}{n}\sum_{i=1}^n Z_i^3 - 3\left(\frac{1}{n}\sum_{i=1}^n Z_i\right)\cdot\left(\frac{1}{n}\sum_{i=1}^n Z_i^2\right) + 2\left(\frac{1}{n}\sum_{i=1}^n Z_i\right)^3,
\end{aligned}$$

and apply delta-method.

(e) (1 pt.) Derive the asymptotic distribution of $\sqrt{n}Y_n$.