Problem Set 3

Econ 212A: Search and Matching Theory

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This problem set is about solving a simple model of asset pricing with transaction costs. The first question studies the benchmark frictionless market with no cost, the second question studies a market with exogenous transaction cost. The third question proposes a simple way to endogenizes the cost of transacting (this is price theory after all!). The setup is a variation on the kind of environment we studied in class: investors receive some utility flow from holding asset which changes over time, creating a need to trade.

Time is continuous and lasts forever. The economy populated by a $[0, 1]$ -continuum of infinitely lived and risk-neutral investors who discount the future at the constant rate $r > 0$. An investor enjoys the consumption of a non-storable numéraire good called cash, with a marginal utility normalized to 1. There is one asset in positive supply $S \in (0, 1)$. An investor can hold either zero or one share of the asset. When holding one share of the asset, he receives a stochastic utility flow $\theta(t)$ per unit of time. At Poisson arrival times with intensity $\delta > 0$, an investor draws a new utility flow according to a probability distribution function $f(\theta)$ over $[\underline{\theta}, \overline{\theta}]$. The Poisson arrival times are independent across investors. Draws of utility flow are independent and identically distributed over time and across investors.

1. We first assume that investors can buy and sell the asset instantly at price p on some frictionless Walrasian market.

   (a) Let V_0 be the continuation value of an investor holding zero share of the asset, and behaving optimally next time he changes utility flow. Similarly, let $V_1(\theta)$ be the continuation value of an investor with current utility flow θ, holding one share of the asset, and behaving optimally next time he changes utility flow. Write Bellman equation for V_0 and $V_1(\theta)$.

   (b) Show that an optimal trading strategy has the cutoff property. That is, there exists some $\theta^* \in [\underline{\theta}, \overline{\theta}]$ such that investors with $\theta \geq \theta^*$ find it optimal to hold one share of the asset and investors with $\theta < \theta^*$ find it optimal to hold zero share.

(c) Define a steady state equilibrium. Show that $p = \theta^*/r$. Find the equation pinning down θ^*.

(d) Calculate the number of transactions per unit of time.

2. We now study an economy with transaction cost. Namely, we assume that, in order to either buy or sell one share trade, an investor has to incur a cost $c \leq 1/2/(r+\delta)$. Let p be the price of the asset, $p_a = p + c$ the ask price, and $p_b = p - c$ the bid price.

(a) Write Bellman equations for V_0 and $V_1(\theta)$.

(b) Show that an optimal trading strategy is characterized by two cutoffs $\theta_a > \theta_b$ such that investors buy the asset when they switch to $\theta \geq \theta_a$ and they sell when they switch to $\theta \leq \theta_b$. Investors who switch to $\theta \in (\theta_b, \theta_a)$ don't change their asset holding.

(c) Calculate the steady state joint distribution of asset holdings and utility flows. *Hint.* It is enough to calculate the following three fractions: the fraction of investors with utility flow $\theta \leq \theta_b$ who hold one share, the fraction of investors with utility flow $\theta \in (\theta_b, \theta_a)$ who hold one share, and the fraction of investors with utility flow $\theta \geq \theta_a$ who hold one share.

(d) Define a steady state equilibrium.

(e) Show that

$$\frac{\theta_a - \theta_b}{r+\delta} = 2c \tag{1}$$

$$p = \frac{\theta_a}{r} - \frac{\theta_a - \theta_b}{2(r+\delta)} - \frac{\delta}{r+\delta}\int_{\theta_b}^{\theta_a} \theta f(\theta)\, d\theta. \tag{2}$$

(f) Show that θ_a solves

$$-\delta(1-S)\left(1 - F(\theta_a)\right) + \delta S F\left(\theta_a - 2c(r+\delta)\right) = 0, \tag{3}$$

where $F(\theta) = \int_{\underline{\theta}}^{\theta} f(z)\, dz$, and argue that the equilibrium is unique. Give an interpretation of this equation in terms of flows of buy and sell orders.

(g) Show that, at $c = 0$, the derivative of the price with respect to c is

$$\frac{\partial p}{\partial c} = 2S\frac{r+\delta}{r} - 1 - \frac{2\delta}{r}\theta^* f(\theta^*) \tag{4}$$

Consider the special case where $[\underline{\theta}, \overline{\theta}] = [0, 1]$ and $f(\theta)$ is a uniform distribution. Derive a condition on exogenous parameter such that $\partial p/\partial c > 0$, i.e. an increase in transaction cost increases the equilibrium price.

3. Question 2 took the transaction cost c to be exogenous. We now endogenize this cost by explicitly specifying the supply and demand of transaction services. Suppose that the Walrasian market is operated by a representative competitive marketmaker. The marketmaker operates a trading technology that can handle $2K$ buy and sell transactions per unit of time, cannot hold any inventory, and takes as given some bid price p_b and a ask price $p_a \geq p_b$.

   (a) Show that, if $K \geq \delta S\,(1 - S)$, then $p_a = p_b = \theta^*/r$. In other words, if the trading capacity K is large enough, then the bid-ask spread is equal to zero.

   (b) Now, when $K < \delta S\,(1 - S)$, $p_a > p_b$. Using the analysis of question 2, calculate the number of buying and selling order per unit of time and show that θ_a and θ_b solve

$$F(\theta_b) = \frac{K}{\delta S} \tag{5}$$

$$F(\theta_a) = 1 - \frac{K}{\delta(1 - S)}. \tag{6}$$

   (c) How does the bid-ask spread $p_a - p_b$ depend on K? Show that the effect of an increase in δ is indeterminate. Explain the two effects that go in opposite direction.