Problem Set 1
Econ 212A: Search and Matching Theory
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1. Time is continuous and runs forever. All agents are risk-neutral and discount the future at the same rate $r > 0$.

   There is a measure one of sellers who can produce some indivisible consumption good at unit cost $c \in [0, 1]$. Assume that production costs are uniformly distributed across sellers. That is, the fraction of sellers with a production cost less than $c \in [0, 1]$ is equal to c. Note that sellers can produce multiple times and stay in the market forever.

   There is a flow F of buyers who enter the market at each time. A buyer's valuation for one unit of good is equal to 1. A buyer seeks to consume one unit of good, and leaves the market after consuming. Assume that a buyer dies at some exogenous Poisson arrival time with intensity $\kappa > 0$.

   Buyers contact sellers at Poisson arrival times with intensity λ. Conditional on a contact, all sellers are equally likely to be contacted. Note that it is assumed that a buyer contacts sellers directly and do not face the risk of contacting another buyer.

   When a buyer and seller meet, the production cost of the seller is observable. The terms of trade are determined according to the Nash solution, assuming that the bargaining power of the buyer is equal to $\theta \in [0, 1]$.

   (a) Let μ_B denote the measure of buyers in the population. Argue that the instantaneous rate of contact between buyers and sellers is equal to

   $$M(\mu_B) = \lambda \mu_B.$$

   From a seller's standpoint, what is the arrival intensity of buyers?

   (b) Let V_B be the maximum attainable utility of a buyer, and $V_S(c)$ be the maximum attainable utility of a seller with production cost $c \in [0, 1]$. Lastly, let $P(c)$ be the price when a buyer meets a seller with production cost c. Write the Bellman equations for V_B, $V_S(c)$, and the equation that determine the price $P(c)$. Make sure that the Bellman equation for $V_S(c)$ reflects the fact that a seller stay in the market forever.

(c) Argue that the gains from trade are positive as long as the production cost is less than some C. Find an equation relating C to V_B.

(d) Write down the equation for the steady-state measure μ_B of buyers. Define an equilibrium.

(e) Solve for the equilibrium value of V_B (Hint: it solves a quadratic equation).

(f) What happens to V_B when λ goes to infinity (Hint: distinguish the case $\theta = 0$ and $\theta \in (0, 1]$?

2. Consider the environment of the previous question with the following modification: there are two types of buyers: high-valuation buyers who derive utility $1 + \delta$ from the good, and low-valuation buyers who derive utility of 1. The flow of high- and low-valuation buyers are denoted by F_H and F_L, respectively. The type of the buyers is not observable by the seller. When a buyer and a seller meet, it is assumed the seller makes a take-it-or-leave-it offer to the buyer.

(a) Let V_{BH} and V_{BL} be the maximum attainable utilities of high- and low-valuation buyers. Argue that, in an equilibrium, there are at most two prices $P_L < P_H$. Express P_H and P_L in terms of V_{BH} and V_{BL}.

(b) Let μ_{BH} and μ_{BL} be the measures of high- and low-valuation buyers. Write the Bellman equation for $V_S(c)$. Show that there are two cutoffs $0 \leq c_L \leq c_H \leq 1$ such that sellers with $c \in [0, c_L]$ quote P_L, sellers with $c \in (c_L, c_H]$ quote P_H, and sellers with $c \in (P_H, 1]$ choose not to sell. Explain how c_L and c_H relate to $(P_H, P_L, \mu_{BH}, \mu_{BL})$.

(c) Given the fractions C_L of sellers who quote P_L, and the fraction $C_H - C_L$ of sellers who quote P_H, and give then prices P_L and P_H, write down the Bellman equations for V_{BH} and V_{BL}.

(d) Write down the equations for the steady state measures μ_{BH} and μ_{BL}.

(e) Define a steady-state equilibrium.

(f) Show that $C_H = 1$. Find the fixed point problem solved by C_L. Solve for a steady-state equilibrium.

(g) Now take μ_{BH} and μ_{BL} as given (i.e., assume that buyers and sellers are replaced by identical types when they die or when they trade). Show that there can be multiple equilibria. It is enough here to provide a numerical example, but be careful to show all equilibria. Explain why there are multiple equilibria, and why endogenizing μ_{BH} and μ_{BL} helps restoring uniqueness.