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1) The problem differs is a specific case of the one described in class for the transition rule $g(A'|A)$ being a two-state Markov chain. θ which replaces A, can take values θ_L or θ_H. It is evolving according to the equation: $distribution'_\theta = distribution_\theta \times P$, where $P = \begin{bmatrix} \pi & 1-\pi \\ 1-\pi & \pi \end{bmatrix}$.

Stationary distribution satisfies: $\begin{bmatrix} a & 1-a \end{bmatrix} \begin{bmatrix} \pi & 1-\pi \\ 1-\pi & \pi \end{bmatrix} = \begin{bmatrix} a & 1-a \end{bmatrix}$

This implies $\begin{bmatrix} (\pi-1)(a-1)+\pi a & -\pi(a-1)-a(\pi-1) \end{bmatrix} = \begin{bmatrix} a & 1-a \end{bmatrix}$, Solution is: $\left[a=\frac{1}{2}\right]$

So, the stationary distribution of θ is $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

2) $\theta'|(\theta=\theta_L) = \begin{cases} \theta_L, & \text{with probability } \pi \\ \theta_H, & \text{with probability } 1-\pi \end{cases}$ $\theta'|(\theta=\theta_H) = \begin{cases} \theta_L, & \text{with probability } 1-\pi \\ \theta_H, & \text{with probability } \pi \end{cases}$

Productivity and workers evolve as:

$x'(x,\theta) = x - \Delta^-(x,\theta) + \Delta^+(x,\theta)$ $\quad n(x,\theta) = x - \Delta^-(x,\theta)$.

Here $\Delta^-(x,\theta)$ is number moving out and $\Delta^+(x,\theta)$ moving in an island in state (x,θ).

There are only two states, and under full information, when agents choose both whether to move out and where to go, the numbers of workers will be set at the margin. So, the amount of workers x in an island in period $t+1$ will be completely determined by the state θ of this island in period t. There won't be divergence in decisions because workers don't know tomorrow's state of the economy they are moving to or the one the stay in, and they just go uniformly to islands where the preferred states are more probable, because those are strictly preferred from their discounted expected utility point of view.

Hence, there will be only two numbers of workers possible $\overline{x}$ and $\underline{x}$, corresponding to the high and low state in previous period. The numbers of workers exactly mimick yesterday's state of the island. So, there are four possible "states" (x,θ) corresponding to the 2x2 combinations of those: $(\underline{x},\theta_L), (\overline{x},\theta_L), (\underline{x},\theta_H), (\overline{x},\theta_H)$.

3) Here are the three cases that can happen:

Case 1: $\Delta^-(\theta,x) > 0, \Delta^+(\theta,x) = 0 \quad \Rightarrow n = x - \Delta^-, \quad x' = x - \Delta^-$

Case 2: $\Delta^+(\theta,x) > 0, \Delta^-(\theta,x) = 0 \quad \Rightarrow n = x, \quad x' = x + \Delta^+$

Case 3: $\Delta^+(\theta,x) = \Delta^-(\theta,x) = 0 \quad \Rightarrow n = x, \quad x' = x$

State $(\underline{x},\theta_L)$ corresponds to $x' = \underline{x} = n$ (case 3). Island stayed in the low state and nobody moved.

State $(\overline{x},\theta_L)$ corresponds to $x' = \overline{x} - \Delta^- = \underline{x} = n$ (case 1). The island went from high to low and some workers moved out.

State $(\underline{x},\theta_H)$ corresponds to $x' = \underline{x} + \Delta^+ = \overline{x}, n = \underline{x}$ (case 1). The island went from low to high and some workers moved in, but don't start working yet.

State $(\overline{x},\theta_H)$ corresponds to $x' = \overline{x} = n$ (case 2). Island stayed in the high state and nobody moved.

The corresponding interstate transition matrix $\Gamma = \begin{bmatrix} \pi & 0 & 1-\pi & 0 \\ \pi & 0 & 1-\pi & 0 \\ 0 & 1-\pi & 0 & \pi \\ 0 & 1-\pi & 0 & \pi \end{bmatrix}$

The corresponding stationary distribution solves $[a,b,c,1-a-b-c]\begin{bmatrix}\pi & 0 & 1-\pi & 0\\ \pi & 0 & 1-\pi & 0\\ 0 & 1-\pi & 0 & \pi\\ 0 & 1-\pi & 0 & \pi\end{bmatrix}=$ $[a,b,c,1-a-b-c]$ Solution is: $\left[a=\frac{\pi}{2},b=\frac{1-\pi}{2},c=\frac{1-\pi}{2},1-a-b-c=\frac{\pi}{2}\right].$

So, the states $\{(\underline{x},\theta_L),(\overline{x},\theta_L),(\underline{x},\theta_H),(\overline{x},\theta_H)\}$ happen with probabilities $\left[\frac{\pi}{2},\frac{1-\pi}{2},\frac{1-\pi}{2},\frac{\pi}{2}\right].$

4) So, in the Bellman equation there is no uncertainty about the employment rate. The Bellman equation doesn't have the max operator: $v(x,\theta)=\theta f'(n(x,\theta))+\beta E[v(x',\theta')|\theta]$. I denote v_i - the value in state i. Then, the Bellman equations are the following:

$v_1=\theta_L f'(\underline{x})+\beta\left(\frac{\pi}{2}v_1+\frac{1-\pi}{2}v_3\right)/\frac{1}{2}$ $\quad v_2=\theta_L f'(\underline{x})+\beta\left(\frac{\pi}{2}v_1+\frac{1-\pi}{2}v_3\right)/\frac{1}{2}$

$v_3=\theta_H f'(\underline{x})+\beta\left(\frac{\pi}{2}v_2+\frac{1-\pi}{2}v_4\right)/\frac{1}{2}$ $\quad v_4=\theta_H f'(\overline{x})+\beta\left(\frac{\pi}{2}v_2+\frac{1-\pi}{2}v_4\right)/\frac{1}{2}$

Obviously, $v_1=v_2$. This is because the state and the employment are the same both when staying in the low productivity and transiting from high to low. I.e. employment is low when productivity is low.

5) Recover that $v_1=v_2=\theta_L f'(\underline{x})+\beta(\pi v_1+(1-\pi)v_3)$

$v_3=\theta_H f'(\underline{x})+\beta((1-\pi)v_2+\pi v_4)$ $\quad v_4=\theta_H f'(\overline{x})+\beta((1-\pi)v_2+\pi v_4)$

Besides, for incentive compatibility on an island staying in state θ_L the workers should not be better off by moving to islands in state θ_H. This is a binding constraint, so, they must be indifferent between staying and moving. This implies $v_1=\beta((1-\pi)v_2+\pi v_4)$. We have five equations in six unknowns and the value functions are determined up to a constant: $v_2=\theta_L f'(\underline{x})+\beta(\pi v_2+(1-\pi)v_3)$ $\quad v_3=\theta_H f'(\underline{x})+\beta((1-\pi)v_2+\pi v_4)$ $\quad v_4=\theta_H f'(\overline{x})+\beta((1-\pi)v_2+\pi v_4)$ $v_2=\beta((1-\pi)v_2+\pi v_4)$.

Therefore, $v_3=\theta_H f'(\underline{x})+v_2 \quad\Rightarrow\quad v_2=\theta_L f'(\underline{x})+\beta(\pi v_2+(1-\pi)(\theta_H f'(\underline{x})+v_2)) \quad\Rightarrow$

$\Rightarrow \quad v_2=\left[\frac{1}{1-\beta}\theta_L+\frac{\beta}{1-\beta}(1-\pi)\theta_H\right]f'(\underline{x}) \quad\Rightarrow$

$\Rightarrow \quad v_4=\theta_H f'(\overline{x})+v_2=\theta_H f'(\overline{x})+\left[\frac{1}{1-\beta}\theta_L+\frac{\beta}{1-\beta}(1-\pi)\theta_H\right]f'(\underline{x}) \quad\Rightarrow$

$(1-\beta(1-\pi))v_2=\beta\pi v_4 \quad\Rightarrow\quad (1-\beta+\beta\pi-\beta\pi)\left[\frac{1}{1-\beta}\theta_L+\frac{\beta}{1-\beta}(1-\pi)\theta_H\right]f'(\underline{x})=\beta\pi\theta_H f'(\overline{x})$

Hence, $[\theta_L+\beta(1-\pi)\theta_H]f'(\underline{x})=\beta\pi\theta_H f'(\overline{x})$. This is a necessary condition.

For existence of equilibrium we also require that $\overline{x}>\underline{x}$.

This implies $f'(\overline{x})<f'(\underline{x})\Leftrightarrow\theta_L+\beta(1-\pi)\theta_H<\beta\pi\theta_H \quad\Leftrightarrow\quad \theta_L<(2\pi-1)\beta\theta_H$.

6) Welfare is equal to aggregate production in this equilibirum:

$$D\cdot\theta f(x)=\left[\frac{\pi}{2},\frac{1-\pi}{2},\frac{1-\pi}{2},\frac{\pi}{2}\right]\begin{bmatrix}\theta_L f(\underline{x})\\ \theta_L f(\underline{x})\\ \theta_H f(\underline{x})\\ \theta_H f(\overline{x})\end{bmatrix}=\frac{1}{2}\theta_L f(\underline{x})+\frac{1-\pi}{2}\theta_H f(\underline{x})+\frac{\pi}{2}\theta_H f(\overline{x})$$

Optimality is over how much $(\underline{x},\overline{x})$ should be chosen subject to constraints. We shall do a variational experiment. Let's take the optimal allocation and redistribute ε workers from high to low: $(\underline{x}-\varepsilon,\overline{x}+\varepsilon)$. Taking ε today from the islands in the low state to the islands in the high state tomorrow and assuming a new steady state from tomorrow on, generates the following aggregate marginal return: $-\frac{1}{2}\theta_L f'(\underline{x})+\frac{\beta}{1-\beta}\left[-\frac{1}{2}\theta_L f'(\underline{x})-\frac{1-\pi}{2}\theta_H f'(\underline{x})+\frac{\pi}{2}\theta_H f(\overline{x})\right]$, which has to be negative. This implies $[\theta_L+\beta(1-\pi)\theta_H]f'(\underline{x})\geq\beta\pi\theta_H f(\overline{x})$.

Redistributing ε in the opposite direction leads to $(\underline{x}+\varepsilon,\overline{x}-\varepsilon)$ and marginal return of:

$\frac{1}{2}\theta_L f'(\underline{x})+\frac{\beta}{1-\beta}\left[\frac{1}{2}\theta_L f'(\underline{x})+\frac{1-\pi}{2}\theta_H f'(\underline{x})-\frac{\pi}{2}\theta_H f(\overline{x})\right]$, which also should be negative.

This implies $[\theta_L+\beta(1-\pi)\theta_H]f'(\underline{x})\leq\beta\pi\theta_H f(\overline{x})$.

Therefore, optimality is attained only under $[\theta_L+\beta(1-\pi)\theta_H]f'(\underline{x})=\beta\pi\theta_H f(\overline{x})$.