Economics 212A, Homework 1 — Anton Cheremukhin

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Exercise 1 *Bargaining Game*

a) The instantaneous rate of contact between buyers and sellers is the rate of meetings λ multiplied by the amounts of buyers and sellers, which are μ_B and 1. From the seller's point of view the arrival rate of buyers is the same number divided by the number of sellers, which is 1. $M(\mu_B) = \lambda\mu_B$

b) Bellman for buyers: $rV_B = \lambda E[1 - P(c) - V_B]$

Bellman for sellers: $rV_S(c) = \lambda\mu_B[P(c) - c]$

Bargaining equation: $P(c) = \arg\max (1 - P(c) - V_B)^\theta (P(c) - c)^{1-\theta}$

c) FOC: $\frac{\theta}{1-P(c)-V_B} = \frac{1-\theta}{P(c)-c} \quad \Rightarrow \quad P(c) = (1-\theta)(1-V_B) + \theta c$

There are gains from trade when $(1-\theta)(1-V_B) + \theta c > c$

I.e. for costs sufficiently small, so that $c < C = 1 - V_B$.

d) The equation for the number of buyers is the following:

$\dot{\mu}_B = f_B - \kappa\mu_B - \lambda\mu_B C \qquad \dot{\mu}_B = 0$ in steady state $\qquad$ So, $\mu_B = \frac{f_B}{\kappa+\lambda C}$.

e) $V_S(c) = \frac{\lambda\mu_B}{r}(1-\theta)(1-V_B-c)$

$V_B = \Pr(c < C)\frac{\lambda}{r}[1 - V_B - E[P(c)|c < C]] =$

$= \frac{\lambda}{r}C\left[1 - V_B - (1-\theta)(1-V_B) - \theta\frac{C}{2}\right] = \frac{\lambda}{r}(1-V_B)^2\frac{\theta}{2}$

$V_B = \frac{r}{\theta\lambda}\left(1 + \frac{\theta\lambda}{r} \pm \sqrt{1 + 2\frac{\theta\lambda}{r}}\right)$ if $\theta \neq 0$ and $V_B = 0$ if $\theta = 0$

f) $V_B = 0$ if $\theta = 0$ and $\lambda \to \infty$

$V_B \to 1$ if $\theta > 0$ and $\lambda \to \infty$

Exercise 2 *Bargaining Game with Asymmetric Information*

a) Now assume there are two types of buyers: low $(u = 1)$ and high $(u = 1 + \delta)$

Since the sellers discriminate using the take it or leave it offer, and there are only two types, seller will use only two different types of contracts. They can require either $P_L(c)$ or $P_H(c)$ (or make no offer).

Reservation value for L: $1 - P_L = V_{BL} \quad \Rightarrow \quad P_L = 1 - V_{BL}$

Reservation value for H: $1 + \delta - P_H = V_{BH} \quad \Rightarrow \quad P_H = 1 + \delta - V_{BH}$

b) Let $\alpha(c)$ be the probability that P_L is asked for, let $\beta(c)$ be the probability that P_H is asked for. Consequently, $1 - \alpha(c) - \beta(c)$ is the probability that no offer is made. If the seller sets a high price than only high buyers accept. If the seller sets a low price than both types accept the offer.

Sellers' Bellman equation: $rV_S(c) = \lambda \max_{\alpha,\beta} E\{\alpha(c)(\mu_{BL} + \mu_{BH})[P_L - c] + \beta(c)\mu_{BH}[P_H - c]\}$

The incentives for $\beta(c) = 1$ are: $\mu_{BH}[P_H - c] > (\mu_{BL} + \mu_{BH})[P_L - c]$ and $P_H > c$

This happens, when $c > P_L - \frac{\mu_{BH}}{\mu_{BL}}(P_H - P_L)$ and $c < P_H$

Similarly, $\alpha(c) = 1$ iff $c < P_L - \frac{\mu_{BH}}{\mu_{BL}}(P_H - P_L)$ and $c < P_L$ $\qquad$ Otherwise, both are zero.

Seller will choose L if $c < P_L$ and $c < P_L - \frac{\mu_{BH}}{\mu_{BL}}(P_H - P_L)$

Seller will choose H if $P_L - \frac{\mu_{BH}}{\mu_{BL}}(P_H - P_L) < c < P_H$

Seller will choose 0 if $c > P_L$ and $c > P_H$

Therefore, $C_H = 1 + \delta - V_{BH} = P_H$ if $P_H \leq 1$, $\qquad C_H = 1$ if $P_H > 1$

$C_L = 1 - V_{BL} + \frac{\mu_{BH}}{\mu_{BL}}(V_{BH} - V_{BL} - \delta) = P_L - \frac{\mu_{BH}}{\mu_{BL}}(P_H - P_L) \leq P_L$

c) Low will only accept low offers. Bellman for buyers of low type:
$V_{BL} = \frac{\lambda}{r} E\left[1 - P_L - V_{BL}\right] = 0$
High will accept both types of offers. Bellman for buyers of high type:
$V_{BH} = \frac{\lambda}{r} E\left[1 + \delta - P\left(c\right) - V_{BH}\right] = \frac{\lambda}{r} C_L \left[1 + \delta - 1 + V_{BL} - V_{BH}\right] +$
$+\frac{\lambda}{r}\left(C_H - C_L\right)\left[1 + \delta - 1 - \delta + V_{BH} - V_{BH}\right] = \frac{\lambda}{r} C_L \left[\delta - V_{BH}\right]$
d) $\dot{\mu}_{BH} = f_{BH} - \kappa \mu_{BH} - \lambda \mu_{BH}\left(C_H - C_L\right)$ $\dot{\mu}_{BH} = 0$ in steady state
$\dot{\mu}_{BL} = f_{BL} - \kappa \mu_{BL} - \lambda \mu_{BL} C_L$ $\dot{\mu}_{BL} = 0$ in steady state
Therefore, measures of buyers of both types are $\mu_{BH} = \frac{f_{BH}}{\kappa + \lambda(C_H - C_L)}$, $\mu_{BL} = \frac{f_{BL}}{\kappa + \lambda C_L}$.
e) SS Equilibirum consists of $\{V_S\left(c\right), V_{BL}, V_{BH}, P_L, P_H, \mu_{BH}, \mu_{BL}, C_L, C_H\}$, s.t.
$V_S\left(c\right) = \frac{\lambda}{r}\left\{\left(\mu_{BL} + \mu_{BH}\right)\left[P_L - c\right]_{c<C_L} + \mu_{BH}\left[P_H - c\right]_{C_L<c<C_H}\right\}$
$V_{BL} = \frac{\lambda}{r} E\left[1 - P_L - V_{BL}\right]$ $V_{BH} = \frac{\lambda}{r} E\left[1 + \delta - P\left(c\right) - V_{BH}\right]$
$P_L = 1 - V_{BL}$ $P_H = 1 + \delta - V_{BH}$ $\mu_{BL} = \frac{f_{BL}}{\kappa + \lambda C_L}$ $\mu_{BH} = \frac{f_{BH}}{\kappa + \lambda(C_H - C_L)}$
$C_L = P_L - \frac{\mu_{BH}}{\mu_{BL}}\left(P_H - P_L\right)$ $C_H = \min\{1, P_H\}$
f) If $P_H < 1$ then $P_L > P_H$.Contradiction. Therefore, $P_H \geq 1$. Hence, $C_H = 1$.
$V_{BL} = 0$ $P_L = 1$ $C_H = 1$ $\mu_{BL} = \frac{f_{BL}}{\kappa + \lambda C_L}$ $\mu_{BH} = \frac{f_{BH}}{\kappa + \lambda(1 - C_L)}$
$V_{BH} = \frac{\lambda}{r} C_L\left[\delta - V_{BH}\right]$ $\Rightarrow V_{BH} = \frac{\frac{\lambda}{r} C_L \delta}{1 + \frac{\lambda}{r} C_L} \Rightarrow P_H = 1 + \delta - V_{BH} = 1 + \frac{\delta}{1 + \frac{\lambda}{r} C_L}$
$C_L = P_L - \frac{\mu_{BH}}{\mu_{BL}}\left(P_H - P_L\right) = 1 - \frac{\mu_{BH}}{\mu_{BL}}\left(P_H - 1\right) = 1 - \frac{\frac{f_{BH}}{\kappa + \lambda(1 - C_L)}}{\frac{f_{BL}}{\kappa + \lambda C_L}} \frac{\delta}{1 + \frac{\lambda}{r} C_L}$
This is a fixed-point problem: $\left(1 - C_L\right)\left(1 + \frac{\lambda}{\kappa}\left(1 - C_L\right)\right) = \frac{f_{BH}}{f_{BL}} \delta \frac{1 + \frac{\lambda}{\kappa} C_L}{1 + \frac{\lambda}{r} C_L}$
When $\frac{f_{BH}}{f_{BL}}\delta$ is not too big:
$\left.\left(1 - C_L\right)\left(1 + \frac{\lambda}{\kappa}\left(1 - C_L\right)\right)\right|_{C_L=0} = \frac{\lambda}{\kappa} + 1 > \left.\frac{f_{BH}}{f_{BL}} \delta \frac{1 + \frac{\lambda}{\kappa} C_L}{1 + \frac{\lambda}{r} C_L}\right|_{C_L=0} = \frac{f_{BH}}{f_{BL}}\delta$
$0 = \left.\left(1 - C_L\right)\left(1 + \frac{\lambda}{\kappa}\left(1 - C_L\right)\right)\right|_{C_L=1} < \left.\frac{f_{BH}}{f_{BL}} \delta \frac{1 + \frac{\lambda}{\kappa} C_L}{1 + \frac{\lambda}{r} C_L}\right|_{C_L=1} = \frac{f_{BH}}{f_{BL}} \delta \frac{1 + \frac{\lambda}{\kappa}}{1 + \frac{\lambda}{r}}$
Therefore, there is a solution in $C_L \in [0, 1]$.
Besides, the LHS is decreasing in C_L while the RHS is increasing in C_L if $r > \kappa$.
In this case the solution is also unique. C_L determines the rest of the variables:
$V_{BH} = \frac{\frac{\lambda}{r} C_L \delta}{1 + \frac{\lambda}{r} C_L}$ $P_H = 1 + \frac{\delta}{1 + \frac{\lambda}{r} C_L}$ $\mu_{BL} = \frac{f_{BL}}{\kappa + \lambda C_L}$ $\mu_{BH} = \frac{f_{BH}}{\kappa + \lambda(1 - C_L)}$
g) If $\frac{\mu_{BH}}{\mu_{BL}} = x$ is given, then there can be two solutions.
$\left(1 - C_L\right) = x \frac{\delta}{1 + \frac{\lambda}{r} C_L} \Rightarrow C_L = \frac{1}{2}\left(1 - \frac{r}{\lambda} \pm \sqrt{\left(1 + \frac{r}{\lambda}\right)^2 - 4\frac{r}{\lambda} x \delta}\right)$
Say, $\frac{r}{\lambda} = y = \frac{1}{4}, \delta = 1/2, x = 3$. (There are more high buyers than low buyers).
$C_L = \left.\frac{1}{2}\left(1 - y \pm \sqrt{\left(1 + y\right)^2 - 4 y x \delta}\right)\right|_{x=3, y=\frac{1}{4}, \delta=1/2} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

Endogenizing $x = \frac{f_{BH}}{f_{BL}} \frac{1 + \frac{\lambda}{\kappa} C_L}{1 + \frac{\lambda}{\kappa}(1 - C_L)}$ helps restore uniqueness because it determines a positive correlation between x and C_L, while here it is positive in one case and negative in the other. Endogenous determination of x chooses in a certain way one of the two solutions.

Set $\frac{f_{BH}}{f_{BL}} = 3$, $\frac{\kappa}{\lambda} = \frac{1}{5} < \frac{r}{\lambda} = \frac{1}{4}$. Then $\left(1 - C_L\right)\left(1 + \frac{\lambda}{\kappa}\left(1 - C_L\right)\right) = \frac{f_{BH}}{f_{BL}} \delta \frac{1 + \frac{\lambda}{\kappa} C_L}{1 + \frac{\lambda}{r} C_L}$ $\Leftrightarrow$
$\Leftrightarrow$ $\left(1 - C_L\right)\left(1 + 5\left(1 - C_L\right)\right) = 3 * \frac{1}{2} \frac{1 + 5C_L}{1 + 4C_L}$ - has a unique solution $C_L = \frac{1}{2} \in [0, 1]$.