Econ203C Homework #5 Anton Cheremukhin

May 24, 2006

Exercise 1 *Asymptotic Properties of GMM*

$\widehat{\theta}_n = \arg\min_{\theta\in\Theta} m_n(\theta)' V_n^{-1} m_n(\theta) \qquad V_n \xrightarrow{p} V \qquad m_n(\theta) = \frac{1}{n}\Sigma\varphi(y_i, x_i, \theta), E_0(\varphi(y_i, x_i, \theta_0)) = 0$

a) Theorem: if $(x_i, y_i) - i.i.d., \Theta - compact, \varphi(y,x,\theta) \in C^0, E_0\left(\sup_{\theta\in\Theta}|\varphi(y,x,\theta)|\right) < \infty$ and $\exists!\theta_0 = \arg\min_{\theta\in\Theta} Q_0(\theta) \overset{ULLN}{\Longrightarrow} \widehat{\theta}_n \xrightarrow{p} \theta_0$.

Proof: if $E_0\left(\sup_{\theta\in\Theta}|\varphi(y,x,\theta)|\right) < \infty$ then $m_n(\theta) \overset{p}{\rightrightarrows} E_0(\varphi(y_i, x_i, \theta))$

Therefore, $m_n(\theta)' V_n^{-1} m_n(\theta) \overset{p}{\rightrightarrows} E_0(\varphi(y_i, x_i, \theta))' V^{-1} E_0(\varphi(y_i, x_i, \theta))$. Hence, $\widehat{\theta}_n \xrightarrow{p} \theta_0$.

b) Theorem: if $(x_i, y_i) - i.i.d., \theta_0 \in \Theta int, \varphi(y,x,\theta) \in C^1, \widehat{\theta}_n \xrightarrow{p} \theta_0, E_0\left[\sup_{\theta\in\Theta}\left|\frac{\partial\varphi(y,x,\theta)}{\partial\theta}\right|\right] < \infty$, $0 < A(\theta_0) = E_0\left[\frac{\partial\varphi(y,x,\theta_0)}{\partial\theta}\right] < \infty, W(\theta_0) = E_0\left[\varphi(y,x,\theta_0)\varphi(y,x,\theta_0)'\right] > 0$,

$\sum \frac{\varphi(y,x,\theta_0)}{\sqrt{n}} \xrightarrow{d} N(0, W(\theta_0)) \overset{CLT}{\Longrightarrow} \sqrt{n}\left(\widehat{\theta}_n - \theta_0\right) \xrightarrow{d} N(0, \Lambda(\theta_0))$

Proof: FOC: $2\left(\sum\frac{\partial\varphi\left(y,x,\widehat{\theta}_n\right)}{\partial\theta}\right) V_n^{-1}\left(\sum\varphi\left(y,x,\widehat{\theta}_n\right)\right) = 0$

Taylor-expansion + mean-value theorem: $\sum\frac{\varphi\left(y,x,\widehat{\theta}_n\right)}{\sqrt{n}} = \sum\frac{\varphi(y,x,\theta_0)}{\sqrt{n}} + \left(\sum\frac{\partial\varphi(y,x,\theta^*)}{\sqrt{n}\partial\theta}\right)\left(\widehat{\theta}_n - \theta_0\right)$

Hence, $\left(\sum\frac{\varphi(y,x,\theta_0)}{\sqrt{n}} + \left(\sum\frac{\partial\varphi(y,x,\theta^*)}{n\partial\theta}\right)\sqrt{n}\left(\widehat{\theta}_n - \theta_0\right)\right) V_n^{-1}\left(\sum\varphi\left(y,x,\widehat{\theta}_n\right)\right) = 0$

If $E_0\left[\sup_{\theta\in\Theta}\left|\frac{\partial\varphi(y,x,\theta)}{\partial\theta}\right|\right] < \infty$ then $\sum\frac{\partial\varphi(y,x,\theta)}{n\partial\theta}\Big|_{\widehat{\theta}_n<\theta^*<\theta_0} \overset{p}{\rightrightarrows} A(\theta_0)$

Therefore, by Slutsky,continuity and CLT, we have the result, where

$\Lambda(\theta_0) = \left(A(\theta_0)V^{-1}A(\theta_0)'\right)^{-1} A(\theta_0)V^{-1}W(\theta_0)V^{-1}A(\theta_0)'\left(A(\theta_0)V^{-1}A(\theta_0)'\right)^{-1}$

Exercise 2 *Probit model*

$y_i^* = z_i'\gamma + u_i \qquad u_i|z_i = N(0, \sigma_v^2) \qquad y_i = \Im(y_i^* > 0)$

a) $Pr(y_i = 1|z_i) = Pr(-u_i < z_i'\gamma|z_i) = \Phi(z_i'\gamma/\sigma_v) = \Phi(z_i'\theta)$, $\theta = \gamma/\sigma_v$. We cannot identify both at the same time, because one of them reflects scale, which in principle cannot be recovered when knowing an indicator only.

b) $L(z_i, y_i, \theta) = \Pi\Phi(z_i'\theta)^{y_i}(1 - \Phi(z_i'\theta))^{1-y_i} \qquad \frac{1}{n}\ln L = \frac{1}{n}\Sigma_{y_i=1}\ln\Phi(z_i'\theta) + \frac{1}{n}\Sigma_{y_i=0}\ln(1 - \Phi(z_i'\theta))$

c) FOC: $\Sigma_{y_i=1}\frac{z_i\varphi\left(z_i'\widehat{\theta}_n\right)}{\Phi\left(z_i'\widehat{\theta}_n\right)} = \Sigma_{y_i=0}\frac{z_i\varphi\left(z_i'\widehat{\theta}_n\right)}{1-\Phi\left(z_i'\widehat{\theta}_n\right)}$

d) The same FOC we shall get when using GMM for the following moment of the data: $E\frac{\partial\ln L(z_i,y_i,\theta)}{\partial\theta} = 0$, which follows from the zero mean of the score function.

Exercise 3 *GMM derivations.*

$y_i = x_i'\beta_0 + \varepsilon_i \qquad E(\varepsilon_i|x_i) = 0 \qquad \varphi_j = z_{ji}(y_i - x_i'\beta)$, e.g. $z_{ji} = x_i \qquad z_{2i} = \left(x_i', x_i^{2\prime}, (x_{1i}x_i)'\right)'$

a) $\beta_0 = (\beta_1, \beta_2, ..., \beta_K) = \frac{\partial E(y|x)}{\partial x'}$.

b) $E_0(\varphi_j(x,y,z,\beta_0)|x) = E[E(\varphi_j(x,y,z,\beta_0)|x)] = E[E((y - x'\beta_0)z_j|x)] =$
$= E[z_j E(\varepsilon|x)] = 0$

c) $\widehat{\beta}_n^j = \arg\min m_n^j(\beta)' \left(\widehat{V}_n^j\right)^{-1} m_n^j(\beta)$

where $\widehat{V}_n^j = \frac{1}{n}\Sigma\varphi_j\left(y_i, x_i, z_j, \widehat{\beta}^j\right)\varphi_j\left(y_i, x_i, z_j, \widehat{\beta}^j\right)', \qquad m_n^j(\beta) = \frac{1}{n}\Sigma\varphi_j(y_i, x_i, \beta)$.

Can use any consistent $\widehat{\beta}^j$, e.g. OLS.

d) Asymptotic distribution: $\sqrt{n}\left(\widehat{\beta}_n^j - \beta_0\right) \xrightarrow{d} N(0, \Lambda_j(\beta_0))$, where

$A_j(\beta_0) = E_0\left[\frac{\partial\varphi_j(y,x,z,\beta_0)}{\partial\beta}\right] = z_j x' \qquad , V_j(\beta_0) = E_0\left[\varphi_j(y,x,z,\beta_0)\varphi_j(y,x,z,\beta_0)'\right]$

$\Lambda_j(\beta_0) = \left(A_j(\beta_0)V_j(\beta_0)^{-1}A_j(\beta_0)'\right)^{-1}$

e) a consistent estimator for $A_j(\beta_0)$ is $\widehat{A}_j = \sum\frac{\partial\varphi_j(y_i,x_i,z_i,\beta_0)}{n\partial\beta} = \sum\left(\frac{z_j x'}{n}\right)_i$

a consistent estimator for $V_j(\beta_0)$ is $\widehat{V}_n^j = \frac{1}{n}\sum z_{ji}z_{ji}'(y_i - x_i'\beta)^2$ - non-singular, p.s.d.

Proof is exactly the same as in exercise 1.

f) $\widehat{\beta}_n^j = \arg\min m_n^j(\beta)' m_n^j(\beta) \qquad$ Then $\sqrt{n}\left(\widehat{\beta}_n^j - \beta_0\right) \xrightarrow{d} N(0, \Lambda_j(\beta_0))$

where $\Lambda_j(\beta_0) = \left(A_j(\beta_0)A_j(\beta_0)'\right)^{-1}\left(A_j(\beta_0)V_j(\beta_0)^{-1}A_j(\beta_0)'\right)\left(A_j(\beta_0)A_j(\beta_0)'\right)^{-1}$

Exercise 4 *GMM empirical.*

Results:

value	OLS	a1	a2	b1	b2	d
β_1	1.01	1.01	1.01	1.17	1.02	1.02
β_2	1.00	1.00	1.00	0.988	0.996	0.996
β_3	-1.02	-1.02	-1.02	-1.05	-1.02	-1.02
β_4	0.49	0.49	0.49	0.487	0.49	0.49
β_5	-0.49	-0.49	-0.49	-0.51	-0.49	-0.49

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st.er.	OLS	a1	a2	b1	b2	d
β_1	0.105	0.428	0.098	0.318	0.097	0.097
β_2	0.024	0.049	0.024	0.034	0.023	0.023
β_3	0.024	0.046	0.023	0.031	0.023	0.023
β_4	0.023	0.042	0.022	0.033	0.022	0.022
β_5	0.025	0.046	0.025	0.034	0.025	0.025

The values are very similar for all methods. The variances are much better for optimal-weight GMM then for GMM with V=I. The use of GMM estimate of beta to compute the weight matrix does not alter the results.

To perform a Wald test of $H_0 : \beta_2^2 + \beta_4^2 = \beta_3^2 + \beta_5^2$ we use delta-method.

The p-value is equal to 0.59. The hypothesis is not rejected.