Economics 203C, Homework #4 Anton Cheremukhin

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Exercise 1

By LLN: $\mathbf{A}_n(\theta_0) = \frac{1}{n}\Sigma \mathbf{s}_n(x_i, y_i, \theta_0) = \frac{1}{n}\Sigma \frac{\partial \ln f(x,y,\theta_0)}{\partial\theta} \xrightarrow{p} E\left(\frac{\partial \ln f(x,y,\theta_0)}{\partial\theta}\right) = 0$

Hence, by Slutsky, $\mathbf{A}_n(\theta_0)\mathbf{A}_n(\theta_0)' \xrightarrow{p} E\left(\frac{\partial \ln f(x,y,\theta_0)}{\partial\theta}\right) E\left(\frac{\partial \ln f(x,y,\theta_0)}{\partial\theta}\right) = 0$

Therefore, it is not a consistent estimator of $I(\theta_0)$.

Exercise 2

$\boxed{f(y)} = \sum_{x=0}^{\infty} \frac{\theta}{x!}(y\beta)^x e^{-y\theta - y\beta} = \theta e^{-y\theta} \sum_{x=0}^{\infty} \frac{e^{-y\beta}}{x!}(y\beta)^x = \boxed{\theta e^{-y\theta}}$ - Exponential of (θ).

$\boxed{f(x|y)} = \frac{f(x,y)}{f(y)} = \boxed{\frac{e^{-y\beta}}{x!}(y\beta)^x}$ - Poisson (β).

$\boxed{f(x)} = \int_0^\infty \left(\frac{(y\beta)^x}{x!} e^{-y\beta}\theta e^{-y\theta}\right) dy = \frac{\theta\beta^x}{x!}\int_0^\infty \left(y^x e^{-(\beta+\theta)y}\right) dy = \frac{\theta\beta^x}{x!}\frac{\Gamma(x+1)}{(\beta+\theta)^{x+1}} = \boxed{\frac{\theta}{\beta+\theta}\left(\frac{\beta}{\beta+\theta}\right)^x}$

Use Gamma distribution: $g(x,\alpha,\gamma) = y^{\alpha-1}\psi^\alpha e^{-\psi x}/\Gamma(\alpha)$ $\quad \int_0^\infty y^{\alpha-1}e^{-\psi x}dy = \frac{\Gamma(\alpha)}{\psi^\alpha}$

$\boxed{f(y|x)} = \frac{f(x,y)}{f(x)} = \frac{\frac{y^x\beta^x}{x!}e^{-y\beta}\theta e^{-y\theta}}{\frac{\theta\beta^x}{(\beta+\theta)^{x+1}}} = \frac{(\beta+\theta)((\beta+\theta)y)^x}{x!}e^{-(\beta+\theta)y} = \boxed{\frac{\lambda(\lambda y)^x}{x!}e^{-\lambda y}}$

$\beta + \theta = \lambda \quad \frac{\theta}{\beta+\theta} = \gamma \quad \int_0^\infty \left(\frac{\lambda(\lambda y)^x}{x!}e^{-\lambda y}\right) dy = \frac{\lambda^{x+1}}{x!}\int_0^\infty \left(y^x e^{-\lambda y}\right) dy = \frac{\lambda^{x+1}}{x!}\frac{\Gamma(x+1)}{\lambda^{x+1}} = \boxed{1}$

$\boxed{Ex} = \int_0^\infty \left(\sum_{x=0}^\infty x\frac{(y\beta)^x}{x!}e^{-y\beta}\theta e^{-y\theta}\right) dy = \beta\int_0^\infty y\theta e^{-y\theta}\,dy = \boxed{\frac{\beta}{\theta}}$

$f(x,y,\beta,\theta) = \frac{\theta e^{-(\beta+\theta)y}(\beta y)^x}{x!} \quad \ln f(x,y,\beta,\theta) = \ln\theta - (\beta+\theta)y + x\ln(\beta y) - \ln x!$

$\frac{\partial \ln f(x,y,\beta,\theta)}{\partial\theta} = \frac{1}{\theta} - y \quad \frac{\partial \ln f(x,y,\beta,\theta)}{\partial\beta} = -y + \frac{x}{\beta} \quad \varphi = [\beta,\theta]$

$\frac{\partial^2 \ln f(x,y,\beta,\theta)}{\partial\theta\partial\theta'} = -\frac{1}{\theta^2} \quad \frac{\partial^2 \ln f(x,y,\beta,\theta)}{\partial\theta\partial\beta'} = 0 \quad \frac{\partial^2 \ln f(x,y,\beta,\theta)}{\partial\beta\partial\beta'} = -\frac{x}{\beta^2}$

$L(x,y,\varphi) = \Sigma \ln f(x_i,y_i,\beta,\theta) \to \max_{\beta,\theta} \quad \Sigma\left(\frac{1}{\theta} - y_i\right) = 0 \quad \Sigma\left(-y_i + \frac{x_i}{\beta}\right) = 0$

$\boxed{\widehat{\theta}_{ML} = \frac{1}{\overline{y}}} \quad \boxed{\widehat{\beta}_{ML} = \frac{\overline{x}}{\overline{y}}} \quad I^{-1}(\beta,\theta) = \left[-E\frac{\partial^2 \ln f(x,y,\varphi)}{\partial\varphi\partial\varphi'}\right]^{-1} = \begin{bmatrix} \frac{Ex}{\beta^2} & 0 \\ 0 & \frac{1}{\theta^2}\end{bmatrix}^{-1} = \begin{bmatrix} \theta\beta & 0 \\ 0 & \theta^2\end{bmatrix}$

Asymptotic distribution: $\boxed{\sqrt{n}\left(\begin{bmatrix}\widehat{\beta} \\ \widehat{\theta}\end{bmatrix} - \begin{bmatrix}\beta \\ \theta\end{bmatrix}\right) \xrightarrow{d} N\left(\begin{bmatrix}0 \\ 0\end{bmatrix}, \begin{bmatrix}\theta\beta & 0 \\ 0 & \theta^2\end{bmatrix}\right)}$

$g(\varphi) = \frac{\theta}{\beta+\theta} = \gamma$ By invariance of ML, $\boxed{\widehat{\gamma}_{ML}} = \widehat{g(\varphi)} = g(\widehat{\varphi}) = \frac{\widehat{\theta}}{\widehat{\beta}+\widehat{\theta}} = \frac{\frac{1}{\overline{y}}}{\frac{\overline{x}}{\overline{y}}+\frac{1}{\overline{y}}} = \boxed{\frac{1}{\overline{x}+1}}$

Use δ-method. Asymptotic distribution: $\sqrt{n}\left(g(\widehat{\varphi}) - g(\varphi)\right) \xrightarrow{d} N\left(0, GI^{-1}(\beta,\theta)G'\right)$

$GI^{-1}(\beta,\theta)G' = \begin{bmatrix} -\frac{\theta}{(\beta+\theta)^2} & \frac{\beta}{(\beta+\theta)^2}\end{bmatrix}\begin{bmatrix}\theta\beta & 0 \\ 0 & \theta^2\end{bmatrix}\begin{bmatrix} -\frac{\theta}{(\theta+\beta)^2} \\ \frac{\beta}{(\theta+\beta)^2}\end{bmatrix} = \frac{\theta^2\beta}{(\theta+\beta)^3}$

$\boxed{\sqrt{n}(\widehat{\gamma} - \gamma) \xrightarrow{d} N\left(0, \frac{\theta^2\beta}{(\theta+\beta)^3}\right)} \quad \boxed{\sqrt{n}\left(\widehat{\beta} - \beta\right) \xrightarrow{d} N(0, \theta\beta)} \quad \boxed{\sqrt{n}\left(\widehat{\theta} - \theta\right) \xrightarrow{d} N\left(0, \theta^2\right)}$

$\lambda = \beta + \theta \quad \boxed{\widehat{\lambda}_{ML}} = \widehat{\beta}_{ML} + \widehat{\theta}_{ML} = \boxed{\frac{\overline{x}}{\overline{y}} + \frac{1}{\overline{y}}} \quad \boxed{\sqrt{n}\left(\widehat{\lambda} - \lambda\right) \xrightarrow{d} N(0, \theta(\theta+\beta))}$

Exercise 3

$E(y_i|x_i) = \Pr(y_i|x_i,\theta) = \Phi(x_i'\gamma) \qquad \frac{\partial E(y_i|x_i)}{\partial x_i} = \varphi(x_i'\gamma_0)\gamma_0$

$\ln L = \Sigma_{i=1}^n [y_i \ln\Phi(x_i'\gamma) + (1-y_i)\ln(1-\Phi(x_i'\gamma))]$

FOC: $\frac{\partial \ln L}{\partial\gamma} = \Sigma_{i=1}^n \left[\frac{x_i y_i}{\Phi(x_i'\gamma)}\varphi(x_i'\gamma) - \frac{(1-y_i)x_i}{1-\Phi(x_i'\gamma)}\varphi(x_i'\gamma)\right] = 0 \qquad \Rightarrow \qquad \widehat{\gamma}_n$

$I(\gamma) = E\left[\frac{\partial \ln L}{\partial\gamma}\frac{\partial \ln L}{\partial\gamma}\right] \qquad \Lambda_0 = I^{-1}(\gamma_0) \qquad \sqrt{n}(\widehat{\gamma}_n - \gamma) \xrightarrow{d} N(0,\Lambda_0)$

Estimators for Λ_0 :

$\widehat{\Lambda}_1 = -\left[\frac{1}{n}\Sigma_{i=1}^n \frac{\partial}{\partial\gamma'}\left[\frac{x_i y_i}{\Phi(x_i'\widehat{\gamma}_n)}\varphi(x_i'\widehat{\gamma}_n) - \frac{(1-y_i)x_i}{1-\Phi(x_i'\widehat{\gamma}_n)}\varphi(x_i'\widehat{\gamma}_n)\right]\right]^{-1}$ - uses Hessian.

$\widehat{\Lambda}_2 = \left[\frac{1}{n}\Sigma_{i=1}^n \left[\frac{x_i y_i}{\Phi(x_i'\widehat{\gamma}_n)}\varphi(x_i'\widehat{\gamma}_n) - \frac{(1-y_i)x_i}{1-\Phi(x_i'\widehat{\gamma}_n)}\varphi(x_i'\widehat{\gamma}_n)\right]\left[\frac{x_i y_i}{\Phi(x_i'\widehat{\gamma}_n)}\varphi(x_i'\widehat{\gamma}_n) - \frac{(1-y_i)x_i}{1-\Phi(x_i'\widehat{\gamma}_n)}\varphi(x_i'\widehat{\gamma}_n)\right]'\right]^{-1}$ - uses OPG. Since ML is consistent, $\widehat{\gamma}_n \xrightarrow{p} \gamma_0$, by LLN and Slutsky:

$\frac{1}{n}\Sigma_{i=1}^n \left[\frac{x_i y_i}{\Phi(x_i'\widehat{\gamma}_n)}\varphi(x_i'\widehat{\gamma}_n) - \frac{(1-y_i)x_i}{1-\Phi(x_i'\widehat{\gamma}_n)}\varphi(x_i'\widehat{\gamma}_n)\right]\left[\frac{x_i y_i}{\Phi(x_i'\widehat{\gamma}_n)}\varphi(x_i'\widehat{\gamma}_n) - \frac{(1-y_i)x_i}{1-\Phi(x_i'\widehat{\gamma}_n)}\varphi(x_i'\widehat{\gamma}_n)\right]' \xrightarrow{p}$

$\frac{1}{n}\Sigma_{i=1}^n \left[\frac{x_i y_i}{\Phi(x_i'\gamma_0)}\varphi(x_i'\gamma_0) - \frac{(1-y_i)x_i}{1-\Phi(x_i'\gamma_0)}\varphi(x_i'\gamma_0)\right]\left[\frac{x_i y_i}{\Phi(x_i'\gamma_0)}\varphi(x_i'\gamma_0) - \frac{(1-y_i)x_i}{1-\Phi(x_i'\gamma_0)}\varphi(x_i'\gamma_0)\right]' \xrightarrow{p}$

$\xrightarrow{p} E\left[\frac{\partial \ln L}{\partial\gamma}\frac{\partial \ln L}{\partial\gamma'}\right]_{\gamma_0} \qquad \frac{1}{n}\Sigma_{i=1}^n \frac{\partial}{\partial\gamma'}\left[\frac{x_i y_i}{\Phi(x_i'\widehat{\gamma}_n)}\varphi(x_i'\widehat{\gamma}_n) - \frac{(1-y_i)x_i}{1-\Phi(x_i'\widehat{\gamma}_n)}\varphi(x_i'\widehat{\gamma}_n)\right] \xrightarrow{p} E\left[\frac{\partial^2 \ln L}{\partial\gamma\partial\gamma'}\right]_{\gamma_0}$

Therefore, by Slutsky, both $\widehat{\Lambda}_2 \xrightarrow{p} \Lambda_0$ and $\widehat{\Lambda}_1 \xrightarrow{p} \Lambda_0$

Exercise 4

$\ln f = y\ln\Phi(x'\gamma) + (1-y)\ln(1-\Phi(x'\gamma)) \qquad \frac{\partial \ln f}{\partial\gamma} = \frac{xy}{\Phi(x'\gamma)}\varphi(x'\gamma) - \frac{(1-y)x}{1-\Phi(x'\gamma)}\varphi(x'\gamma)$

That's equivalent to exercise 3. Numerical solution using Matlab:

	β_1	β_2	β_3	β_4
coef	0.8235	0.9288	-0.8624	0.3940
s.e.	0.2267	0.1221	0.1248	0.0933
t-stat	3.6317	7.6083	-6.9096	4.2251
p-val	0.0003	0.0000	0.0000	0.0000

To test a hypothesis $\beta_2 + \beta_3 = 1$ we use Wald test: $(\Gamma\widehat{\gamma})'\left(\Gamma\widehat{\Lambda}_2\Gamma'\right)^{-1}(\Gamma\widehat{\gamma}) \sim \chi^2(1)$

$\Gamma = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$ P-value is 0.372. The hypothesis is not rejected.

Elasticities: $\epsilon_{y,x_i} = \frac{\partial E(y_i|x_i)}{\partial x_i}\frac{\overline{x}}{\overline{y}} = \varphi(x_i'\gamma_0)\gamma_0\frac{\overline{x}}{\overline{y}}$

y	x_1	x_2	x_3	x_4
elasticity	0.0978	0.2293	-0.2114	0.0905
s.e.	0.1114	0.1195	0.1235	0.0861

Because of invariance of MLE $\widehat{h(\beta)} = h\left(\widehat{\beta}\right) = \frac{\widehat{\beta}_2\widehat{\beta}_1}{\widehat{\beta}_3^2} = 1.0283.$

To find a distribution of $h(\beta) = \frac{\beta_2\beta_1}{\beta_3^2}$ we use δ-method: $\sqrt{n}\left(\widehat{\beta} - \beta\right) \xrightarrow{d} N(0,\Lambda_0)$.

$\sqrt{n}\left(h\left(\widehat{\beta}\right) - h(\beta)\right) \xrightarrow{d} N(0,\Gamma\Lambda_0\Gamma')$ where we evaluate $\left(\Gamma\widehat{\Lambda}_2\Gamma'\right)$ using $\Gamma = \begin{bmatrix} \frac{\beta_2}{\beta_3^2} & \frac{\beta_1}{\beta_3^2} & -2\frac{\beta_2\beta_1}{\beta_3^3} & 0 \end{bmatrix}$.
Standard error of the estimate is equal 0.198.