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Exercise 1

1) $\widehat{\beta_{IV}} = \left(\widehat{\Pi}'Z'X\right)^{-1}\left(\widehat{\Pi}'Z'y\right) \qquad \widehat{\Pi} \xrightarrow{p} \Pi_0$

$\sqrt{n}\left(\widehat{\beta_{IV}} - \beta\right) = \sqrt{n}\left(\widehat{\Pi}'Z'X\right)^{-1}\left(\widehat{\Pi}'Z'\varepsilon\right) = \left(\frac{1}{n}\widehat{\Pi}'Z'X\right)^{-1}\left(\frac{1}{\sqrt{n}}\widehat{\Pi}'Z'\varepsilon\right)$

By CLT: $\frac{1}{\sqrt{n}}Z'\varepsilon \xrightarrow{d} N(0, V_0)$ By Slutsky: $\frac{1}{\sqrt{n}}\widehat{\Pi}'Z'\varepsilon \xrightarrow{d} N(0, \Pi_0' V_0 \Pi_0)$

By WLLN: $\frac{1}{n}Z'X \xrightarrow{p} \Sigma_{zx}$ By Slutsky and Mann-Wald: $\left(\frac{1}{n}\widehat{\Pi}'Z'X\right)^{-1} \xrightarrow{p} (\Pi_0'\Sigma_{zx})^{-1}$

By Slutsky: $\sqrt{n}\left(\widehat{\beta_{IV}} - \beta\right) \xrightarrow{d} N\left(0, (\Pi_0'\Sigma_{zx})^{-1}\Pi_0'V_0\Pi_0(\Sigma_{zx}'\Pi_0)^{-1}\right) = N(0, \Lambda_0)$

2) $\Pi_0 = V_0^{-1}\Sigma_{zx} \Rightarrow \qquad \Lambda^* = \left(\left(V_0^{-1}\Sigma_{zx}\right)'\Sigma_{zx}\right)^{-1}\left(V_0^{-1}\Sigma_{zx}\right)'V_0\left(V_0^{-1}\Sigma_{zx}\right)\left(\Sigma_{zx}'\left(V_0^{-1}\Sigma_{zx}\right)\right)^{-1} =$

$= \left(\Sigma_{zx}'V_0^{-1}\Sigma_{zx}\right)^{-1}\Sigma_{zx}'V_0^{-1}V_0V_0^{-1}\Sigma_{zx}\left(\Sigma_{zx}'V_0^{-1}\Sigma_{zx}\right)^{-1} = \left(\Sigma_{zx}'V_0^{-1}\Sigma_{zx}\right)^{-1}$

3) $\Pi_0' = \Sigma_{zx}'\Phi^{-1} \qquad \Pi_0 = \Phi'^{-1}\Sigma_{zx} \qquad \Lambda_0 \geq \Lambda^*$ since:

$\Lambda^{*-1} - \Lambda_0^{-1} = \Sigma_{zx}'V_0^{-1}\Sigma_{zx} - \left((\Pi_0'\Sigma_{zx})^{-1}\Pi_0'V_0\Pi_0(\Sigma_{zx}'\Pi_0)^{-1}\right)^{-1} =$

$= \Sigma_{zx}'V_0^{-1}\Sigma_{zx} - \left((\Sigma_{zx}'\Phi^{-1}\Sigma_{zx})^{-1}\Sigma_{zx}'\Phi^{-1}V_0\Phi'^{-1}\Sigma_{zx}(\Sigma_{zx}'\Phi'^{-1}\Sigma_{zx})^{-1}\right)^{-1} =$

$= \Sigma_{zx}'V_0^{-1}\Sigma_{zx} - \Sigma_{zx}'\Phi'^{-1}\Sigma_{zx}(\Sigma_{zx}'\Phi^{-1}V_0\Phi'^{-1}\Sigma_{zx})^{-1}\Sigma_{zx}'\Phi^{-1}\Sigma_{zx} =$

$= \Sigma_{zx}'V_0^{-1/2}\left[I - V_0^{1/2}\Phi'^{-1}\Sigma_{zx}(\Sigma_{zx}'\Phi^{-1}V_0\Phi'^{-1}\Sigma_{zx})^{-1}\Sigma_{zx}'\Phi^{-1}V_0^{1/2}\right]V_0^{-1/2}\Sigma_{zx} =$

$= H[I-Q]H' = HPH' = HPP'H' \geq 0$ because both P and Q are idempotent: $Q = Q'$ and $Q'Q = V_0^{1/2}\Phi'^{-1}\Sigma_{zx}(\Sigma_{zx}'\Phi^{-1}V_0\Phi'^{-1}\Sigma_{zx})^{-1}\Sigma_{zx}'\Phi^{-1}V_0\Phi'^{-1}\Sigma_{zx}(\Sigma_{zx}'\Phi^{-1}V_0\Phi'^{-1}\Sigma_{zx})^{-1}\Sigma_{zx}'\Phi^{-1}V_0^{1/2} = Q$

Exercise 2

Results:

	s.e. OLS	OLS	GLS	s.e. GLS	GLS R
β_{11}	0.0465	0.9897	0.9877	0.0911	0.9879
β_{12}	0.0448	1.0921	1.0922	0.0877	1.1036
β_{13}	0.0466	0.9505	0.9480	0.0912	0.9428
β_{14}	0.0455	0.9618	0.9674	0.0889	0.9637
β_{25}	0.0430	0.9872	0.9846	0.0829	0.9865
β_{26}	0.0441	0.9262	0.9268	0.0844	0.9282
β_{27}	0.0421	0.9983	1.0051	0.0708	1.0018
β_{28}	0.0428	1.0360	1.0319	0.0721	1.0308
β_{31}	0.0851	0.2735	0.9804	0.1413	0.9879
β_{32}	0.0817	0.3698	1.1250	0.1364	1.1036
β_{35}	0.0836	1.1514	1.0395	0.1593	0.9865
β_{36}	0.0847	1.1980	0.8995	0.1607	0.9282
β_{39}	0.0791	0.3359	1.0428	0.1316	1.0532
β_{310}	0.0823	0.3900	0.9769	0.1366	0.9915

Matrix Σ

$$\begin{bmatrix} 3.725 & 0.082 & -0.169 \\ 0.082 & 3.556 & 3.486 \\ -0.169 & 3.486 & 12.10 \end{bmatrix}$$

For the third equation OLS gives bad estimates. OLS underestimates the standard error by a factor of 2. The null of $\beta_{11} = \beta_{31}, \beta_{12} = \beta_{32}, \beta_{25} = \beta_{35}, \beta_{26} = \beta_{36}$ is the easiest to test using and F-test for the unrestricted GLS model. It gives a value of 0.0567 and a p-value of 0.994. So the null is not rejected. To estimate the restricted model it is easiest to use the formula: $\widehat{\beta}_R = \widehat{\beta}_U - (X'\Sigma^{-1}X)^{-1}\Gamma'\left(\Gamma\left(X'\Sigma^{-1}X\right)^{-1}\Gamma'\right)^{-1}\left(\Gamma\widehat{\beta}_U - \Gamma\beta\right)$

Exercise 3

1) $\begin{cases} y_{1i} = \gamma_1 y_{2i} + \beta_{11}x_{1i} + \varepsilon_{1i} \\ y_{2i} = \gamma_2 y_{1i} + \beta_{22}x_{2i} + \beta_{32}x_{3i} + \varepsilon_{2i} \end{cases}$ $\qquad \Sigma z_i z_j = \begin{matrix} & y_1 & y_2 & x_1 & x_2 & x_3 \\ y_1 & 20 & 6 & 4 & 3 & 5 \\ y_2 & 6 & 10 & 3 & 6 & 7 \\ x_1 & 4 & 3 & 5 & 2 & 3 \\ x_2 & 3 & 6 & 2 & 10 & 8 \\ x_3 & 5 & 7 & 3 & 8 & 15 \end{matrix}$

$$\begin{bmatrix} \gamma_1 \\ \beta_{11} \end{bmatrix} = \begin{bmatrix} \Sigma y_2^2 & \Sigma x_1 y_2 \\ \Sigma x_1 y_2 & \Sigma x_1^2 \end{bmatrix}^{-1} \begin{bmatrix} \Sigma y_2 y_1 \\ \Sigma x_1 y_1 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{18}{41} \\ \frac{22}{41} \end{bmatrix} \approx \begin{bmatrix} 0.44 \\ 0.54 \end{bmatrix}$$

$$\begin{bmatrix} \gamma_2 \\ \beta_{22} \\ \beta_{32} \end{bmatrix} = \begin{bmatrix} 20 & 3 & 5 \\ 3 & 10 & 8 \\ 5 & 8 & 15 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{304}{1575} \\ \frac{121}{315} \\ \frac{311}{1575} \end{bmatrix} \approx \begin{bmatrix} 0.19 \\ 0.38 \\ 0.20 \end{bmatrix}$$

2) Instruments: $Z = [x_1, x_2, x_3]$, $\beta_{2SLS} = \left((Z'X)'(Z'Z)^{-1}(Z'X)\right)^{-1}(Z'X)'(Z'Z)^{-1}Z'y = \Psi Z'y$

$$Z'X_1 = \begin{bmatrix} 3 & 5 \\ 6 & 2 \\ 7 & 3 \end{bmatrix}, Z'y_1 = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, Z'Z = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{bmatrix}, Z'X_2 = \begin{bmatrix} 4 & 2 & 3 \\ 3 & 10 & 8 \\ 5 & 8 & 15 \end{bmatrix}, Z'y_2 = \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix}$$

$$\Psi_1 = \begin{bmatrix} -\frac{127}{1334} & \frac{175}{1334} & \frac{95}{1334} \\ \frac{343}{1334} & -\frac{105}{1334} & -\frac{57}{1334} \end{bmatrix} \qquad \begin{bmatrix} \gamma_1 \\ \beta_{11} \end{bmatrix} = \begin{bmatrix} \frac{246}{667} \\ \frac{386}{667} \end{bmatrix} \approx \begin{bmatrix} 0.37 \\ 0.58 \end{bmatrix}$$

$$\Psi_2 = \begin{bmatrix} \frac{43}{128} & -\frac{3}{128} & -\frac{7}{128} \\ -\frac{5}{256} & \frac{45}{256} & -\frac{23}{256} \\ -\frac{13}{128} & -\frac{11}{128} & \frac{17}{128} \end{bmatrix} \qquad \begin{bmatrix} \gamma_2 \\ \beta_{22} \\ \beta_{32} \end{bmatrix} = \begin{bmatrix} \frac{31}{64} \\ \frac{47}{128} \\ \frac{7}{64} \end{bmatrix} \approx \begin{bmatrix} 0.48 \\ 0.37 \\ 0.11 \end{bmatrix}$$

$$V\left(\widehat{\beta}\right) = \Psi Z'Z\Psi'\widehat{\sigma}^2 = \left((Z'X)'(Z'Z)^{-1}(Z'X)\right)^{-1}\widehat{\sigma}^2$$

$$\widehat{\sigma}^2(n-k) = e'e = \left(y - X\widehat{\beta}\right)'\left(y - X\widehat{\beta}\right) = y'y - \widehat{\beta}'X'y - y'X\widehat{\beta} + \widehat{\beta}'X'X\widehat{\beta}$$

$$\sigma_1^2 = \left(20 - 2 * \frac{3020}{667} + \frac{1919876}{444889}\right)/(25-2) = \frac{6788976}{10232447} \approx 0.663$$

$$\sigma_2^2 = \left(10 - 2 * \frac{47}{8} + \frac{69301}{8192}\right)/(25-3) = \frac{54965}{180224} \approx 0.305$$

$$V\left(\begin{bmatrix} \gamma_1 \\ \beta_{11} \end{bmatrix}\right) = \begin{bmatrix} \frac{235}{667} & -\frac{141}{667} \\ -\frac{141}{667} & \frac{218}{667} \end{bmatrix} \frac{6788976}{10232447} \approx \begin{bmatrix} 0.234 & -0.140 \\ -0.140 & 0.217 \end{bmatrix}$$

$$V\left(\begin{bmatrix} \gamma_2 \\ \beta_{22} \\ \beta_{32} \end{bmatrix}\right) = \begin{bmatrix} \frac{2021}{4096} & -\frac{235}{8192} & -\frac{611}{4096} \\ -\frac{235}{8192} & \frac{2885}{16384} & -\frac{691}{8192} \\ -\frac{611}{4096} & -\frac{691}{8192} & \frac{661}{4096} \end{bmatrix} \frac{54965}{180224} \approx \begin{bmatrix} 0.1505 & -0.0087 & -0.0455 \\ -0.0087 & 0.0537 & -0.0257 \\ -0.0455 & -0.0257 & 0.0492 \end{bmatrix}$$

4) $e_1'e_2 = \left(y_1 - X_1\widehat{\beta_1}\right)'\left(y_2 - X_2\widehat{\beta_2}\right) = y_1'y_2 - \widehat{\beta_1}'X_1'y_2 - y_1'X_2\widehat{\beta_2} + \widehat{\beta_1}'X_1'X_2\widehat{\beta_2} =$

$$\left(6 - \begin{bmatrix} \frac{246}{667} & \frac{386}{667} \end{bmatrix} \begin{bmatrix} 10 \\ 3 \end{bmatrix} - \begin{bmatrix} 20 & 3 & 5 \end{bmatrix} \begin{bmatrix} \frac{31}{64} \\ \frac{47}{128} \\ \frac{7}{64} \end{bmatrix} + \begin{bmatrix} \frac{246}{667} & \frac{386}{667} \end{bmatrix} \begin{bmatrix} 6 & 6 & 7 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{31}{64} \\ \frac{47}{128} \\ \frac{7}{64} \end{bmatrix}\right)/25 =$$

$$= -\frac{585449}{2134400} \approx -0.27429 \qquad \widehat{\Sigma} = \left[\frac{1}{n}e_i'e_j\right] = \begin{bmatrix} \frac{6788976}{11122225} & -\frac{585449}{2134400} \\ -\frac{585449}{2134400} & \frac{10993}{40960} \end{bmatrix} \approx \begin{bmatrix} 0.610 & -0.274 \\ -0.274 & 0.2684 \end{bmatrix}$$

$$\widehat{\Sigma}^{-1} = \begin{bmatrix} \frac{1222666194250}{403561600079} & \frac{1249582345600}{403561600079} \\ \frac{1249582345600}{403561600079} & \frac{2780764569600}{403561600079} \end{bmatrix} \qquad X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \mathbb{Z} = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}$$

$$X_1' Z \left(Z'Z\right)^{-1} Z' y_1 = \begin{bmatrix} \frac{162}{47} \\ 4 \end{bmatrix} \qquad X_2' Z \left(Z'Z\right)^{-1} Z' y_2 = \begin{bmatrix} \frac{162}{47} \\ 6 \\ 7 \end{bmatrix} \qquad X_1' Z \left(Z'Z\right)^{-1} Z' y_2 = \begin{bmatrix} \frac{218}{47} \\ 3 \end{bmatrix}$$

$$X_2' Z \left(Z'Z\right)^{-1} Z' y_1 = \begin{bmatrix} \frac{349}{94} \\ 3 \\ 5 \end{bmatrix} \qquad X_1' Z \left(Z'Z\right)^{-1} Z' X_1 = \begin{bmatrix} \frac{218}{47} & 3 \\ 3 & 5 \end{bmatrix} \qquad X_1' Z \left(Z'Z\right)^{-1} Z' X_2 =$$

$$\begin{bmatrix} \frac{162}{47} & 6 & 7 \\ 4 & 2 & 3 \end{bmatrix} \qquad X_2' Z \left(Z'Z\right)^{-1} Z' X_1 = \begin{bmatrix} \frac{162}{47} & 4 \\ 6 & 2 \\ 7 & 3 \end{bmatrix} \qquad X_2' Z \left(Z'Z\right)^{-1} Z' X_2 = \begin{bmatrix} \frac{349}{94} & 3 & 5 \\ 3 & 10 & 8 \\ 5 & 8 & 15 \end{bmatrix}$$

$$Z \left(Z'Z\right)^{-1} Z' = \Phi \qquad \beta_{3SLS} = \begin{bmatrix} \Sigma_{11}^{-1} X_1' \Phi X_1 & \Sigma_{21}^{-1} X_1' \Phi X_2 \\ \Sigma_{12}^{-1} X_2' \Phi X_1 & \Sigma_{21}^{-1} X_2' \Phi X_2 \end{bmatrix}^{-1} \begin{bmatrix} \Sigma_{11}^{-1} X_1' \Phi y_1 + \Sigma_{12}^{-1} X_1' \Phi y_2 \\ \Sigma_{21}^{-1} X_2' \Phi y_1 + \Sigma_{22}^{-1} X_2' \Phi y_2 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1222666194250}{403561600079} \begin{bmatrix} \frac{218}{47} & 3 \\ 3 & 5 \end{bmatrix} & \frac{1249582345600}{403561600079} \begin{bmatrix} \frac{162}{47} & 6 & 7 \\ 4 & 2 & 3 \end{bmatrix} \\ \frac{1249582345600}{403561600079} \begin{bmatrix} \frac{162}{47} & 4 \\ 6 & 2 \\ 7 & 3 \end{bmatrix} & \frac{2780764569600}{403561600079} \begin{bmatrix} \frac{349}{94} & 3 & 5 \\ 3 & 10 & 8 \\ 5 & 8 & 15 \end{bmatrix} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1222666194250}{403561600079} \begin{bmatrix} \frac{162}{47} \\ 4 \end{bmatrix} + \frac{1249582345600}{403561600079} \begin{bmatrix} \frac{218}{47} \\ 3 \end{bmatrix} \\ \frac{2780764569600}{403561600079} \begin{bmatrix} \frac{162}{47} \\ 6 \\ 7 \end{bmatrix} + \frac{1249582345600}{403561600079} \begin{bmatrix} \frac{349}{94} \\ 3 \\ 5 \end{bmatrix} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{266541230346500}{18967395203713} & \frac{3667998582750}{403561600079} & \frac{202432339987200}{18967395203713} & \frac{7497494073600}{403561600079} & \frac{8747076419200}{403561600079} \\ \frac{3667998582750}{403561600079} & \frac{6113330971250}{403561600079} & \frac{4998329382400}{403561600079} & \frac{2499164691200}{403561600079} & \frac{3748747036800}{403561600079} \\ \frac{202432339987200}{18967395203713} & \frac{4998329382400}{403561600079} & \frac{485243417395200}{18967395203713} & \frac{8342293708800}{403561600079} & \frac{13903822848000}{403561600079} \\ \frac{7497494073600}{403561600079} & \frac{2499164691200}{403561600079} & \frac{8342293708800}{403561600079} & \frac{27807645696000}{403561600079} & \frac{22246116556800}{403561600079} \\ \frac{8747076419200}{403561600079} & \frac{3748747036800}{403561600079} & \frac{13903822848000}{403561600079} & \frac{22246116556800}{403561600079} & \frac{41711468544000}{403561600079} \end{bmatrix}^{-1} \begin{bmatrix} \frac{470480874809300}{18967395203713} \\ \frac{8639411813800}{403561600079} \\ \frac{668535979582400}{18967395203713} \\ \frac{20433334454400}{403561600079} \\ \frac{25713263715200}{403561600079} \end{bmatrix} = \begin{bmatrix} \frac{246}{667} \\ \frac{386}{667} \\ \frac{13111619045}{27807645696} \\ \frac{17262742453}{55615291392} \\ \frac{22742485}{138346496} \end{bmatrix} \approx \begin{bmatrix} 0.36882 \\ 0.57871 \\ 0.47151 \\ 0.31040 \\ 0.16439 \end{bmatrix}$$

5)
$$\begin{aligned} y_{1i} &= \gamma_1 y_{2i} + \beta_{11} x_{1i} + \varepsilon_{1i} \\ y_{2i} &= \gamma_2 y_{1i} + \beta_{22} x_{2i} + \beta_{32} x_{3i} + \varepsilon_{2i} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} y_{1i} &= \tfrac{\beta_{11}}{1-\gamma_1\gamma_2} x_{1i} + \tfrac{\gamma_1\beta_{22}}{1-\gamma_1\gamma_2} x_{2i} + \tfrac{\gamma_1\beta_{32}}{1-\gamma_1\gamma_2} x_{3i} + \tfrac{\gamma_1\varepsilon_{2i}+\varepsilon_{1i}}{1-\gamma_1\gamma_2} \\ y_{2i} &= \tfrac{\gamma_2\beta_{11}}{1-\gamma_1\gamma_2} x_{1i} + \tfrac{\beta_{22}}{1-\gamma_1\gamma_2} x_{2i} + \tfrac{\beta_{32}}{1-\gamma_1\gamma_2} x_{3i} + \tfrac{\gamma_2\varepsilon_{1i}+\varepsilon_{2i}}{1-\gamma_1\gamma_2} \end{aligned}$$

$$\begin{bmatrix} \frac{\beta_{11}}{1-\gamma_1\gamma_2} \\ \frac{\gamma_1\beta_{22}}{1-\gamma_1\gamma_2} \\ \frac{\gamma_1\beta_{32}}{1-\gamma_1\gamma_2} \\ \frac{\gamma_2\beta_{11}}{1-\gamma_1\gamma_2} \\ \frac{\beta_{22}}{1-\gamma_1\gamma_2} \\ \frac{\beta_{32}}{1-\gamma_1\gamma_2} \end{bmatrix} = \begin{bmatrix} \left(Z'Z\right)^{-1} Z' y_1 \\ \left(Z'Z\right)^{-1} Z' y_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} \\ \begin{bmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{32}{47} \\ \frac{1}{94} \\ \frac{9}{47} \\ \frac{31}{94} \\ \frac{35}{94} \\ \frac{19}{94} \end{bmatrix} \approx \begin{bmatrix} 0.68085 \\ 0.01064 \\ 0.19149 \\ 0.32979 \\ 0.37234 \\ 0.20213 \end{bmatrix}$$

$\gamma_1 = \frac{1}{94} / \frac{35}{94} = \frac{1}{35} \neq \frac{9}{47} / \frac{19}{94} = \frac{18}{19} \qquad \gamma_2 = \frac{31}{94} / \frac{32}{47} = \frac{31}{64}$

The first equation is overidentified. The second is exactly identified.

Can solve for β_{22} and β_{32} : $y_{2i} - \gamma_2 y_{1i} = \beta_{22} x_{2i} + \beta_{32} x_{3i} + \varepsilon_{2i}$

$$\begin{bmatrix} \beta_{22} \\ \beta_{32} \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 8 & 15 \end{bmatrix}^{-1} \left(\begin{bmatrix} 6 \\ 7 \end{bmatrix} - \frac{31}{64} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} \frac{47}{128} \\ \frac{7}{64} \end{bmatrix} \approx \begin{bmatrix} 0.367 \\ 0.109 \end{bmatrix}$$