Econ203C Homework #1 Anton Cheremukhin

April 12, 2006

Exercise 1

$c_i = \beta_1 + \beta_2 y_i^* + u_i \qquad u_i \sim i.i.d.\,(0,\sigma^2) \qquad$ observe $y_i = y_i^* + v_i \qquad v_i \sim i.i.d.\,(0,w^2)$
$E(v_i y_i^*) = E(v_i u_i) = 0 \qquad$ regress $\qquad c_i = \beta_1 + \beta_2 y_i + \varepsilon_i$
a) $\beta_1 + \beta_2 y_i^* + u_i = c_i = \beta_1 + \beta_2 y_i + \varepsilon_i = \beta_1 + \beta_2(y_i^* + v_i) + \varepsilon_i = \beta_1 + \beta_2 y_i^* + (\beta_2 v_i + \varepsilon_i)$
$\varepsilon_i = u_i - \beta_2 v_i \qquad y_i = y_i^* + v_i \qquad cov(y_i, \varepsilon_i) = E[(y_i^* + v_i - Ey_i^*)(u_i - \beta_2 v_i)] =$
$= Ey_i^* u_i - \beta_2 E y_i^* v_i + E v_i u_i - \beta_2 E v_i^2 - E y_i^*(E u_i - \beta_2 E v_i) = -\beta_2 E v_i^2 < 0$ if $\beta_2 > 0$
b) $\widehat{\beta}_2 = \left(\frac{1}{n}\Sigma(y_i - \overline{y_i})^2\right)^{-1}\left(\frac{1}{n}\Sigma(y_i - \overline{y_i})(c_i - \overline{c_i})\right) = \beta_2 + \left(\frac{1}{n}\Sigma(y_i - \overline{y_i})^2\right)^{-1}\left(\frac{1}{n}\Sigma(y_i - \overline{y_i})\varepsilon_i\right)$
$\left(\frac{1}{n}\Sigma(y_i - \overline{y_i})^2\right)^{-1} \xrightarrow{p} var(y_i) \qquad \left(\frac{1}{n}\Sigma(y_i - \overline{y_i})\varepsilon_i\right) \xrightarrow{p} cov(y_i, \varepsilon_i) = -\beta_2 E v_i^2$
$\widehat{\beta}_2 \xrightarrow{p} \beta_2\left(1 - \frac{w^2}{var(y_i^*) + w^2}\right) < \beta_2$

Exercise 2

$y_i = x_i\beta_0 + \sigma_u u_i \qquad E(u_i v_i) = \rho \qquad u_i \sim N(0,1) \qquad v_i \sim N(0,1) \qquad i = 1,n$
$x_i = w_i\pi_0 + \sigma_v v_i \qquad \exists w_i: \qquad w_i' w_i = 1, \qquad cov(w_i, v_i) = 0$

a) Though not stated clearly enough in the problem, we need to assume a joint normal distribution. Otherwise we can generate $u_i = \{v_i \text{ if } |v_i| < a; -v_i \text{ if } |v_i| \geq a\}$, and construct $\varepsilon_i = u_i - \rho v_i$ such that both are normal, $E(u_i v_i) = \rho$, but $\varepsilon_i | v_i$ is a known non-zero function.

$\begin{bmatrix} u_i \\ v_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right) \Rightarrow u_i|v_i \sim N(0 + \rho(v_i - 0), 1 - \rho^2) = N(\rho v_i, 1 - \rho^2)$

Hence, $E(u_i|v_i) = \rho v_i, \qquad u_i|v_i = \rho v_i + N(0, 1-\rho^2) = \rho v_i + \varepsilon_i|v_i$

Define $\varepsilon_i = u_i - \rho v_i$. Then, $u_i|v_i = \rho v_i + \varepsilon_i|v_i. \qquad E(\varepsilon_i|v_i) = E(u_i|v_i) - \rho v_i = 0$

b) Since w is an instrument, $\widehat{\beta}_{IV} = (wx')^{-1}(wy)$

c) $\widehat{\beta}_{IV} - \beta_0 = \frac{w(x'\beta_0 + \sigma_u u)}{wx'} - \beta_0 = \frac{\sigma_u w u}{wx'} = \frac{\sigma_u w(\rho v + \varepsilon)}{w(w'\pi + \sigma_v v)'} = \frac{\sigma_u(\rho w v + w\varepsilon)}{\pi + \sigma_v w v'}$

d) if $\sigma_v = 0$ then $E\left(\widehat{\beta}_{IV} - \beta_0\right) = E\frac{\sigma_u(\rho w v + w\varepsilon)}{\pi} = \frac{\sigma_u}{\pi}(\rho E w v + E w\varepsilon) = 0.$

Intuition: if there is no uncertainty in the second equation, then x is not stochastic and by substituting equations the system can shown to be equivalent to standard CLR (by the way, assumption $\sigma_v = 0$ violates $E(u_i v_i) = \rho$ unless it is a zero constant).

Exercise 3

a) To be an instrument it must be true that $E(zu) = 0$ and $cov(z,x) \neq 0$

b) $\widehat{\beta}_{IV} - \beta = \left(\left(\frac{1}{n}X'Z\right)A^{-1}\left(\frac{1}{n}Z'X\right)\right)^{-1}\left(\left(\frac{1}{n}X'Z\right)A^{-1}\left(\frac{1}{n}Z'u\right)\right) \xrightarrow{p} 0$ since
$\left(\frac{1}{n}Z'u\right) \xrightarrow{p} E(zu) = 0$ and $\left(\frac{1}{n}Z'X\right) \xrightarrow{p} \Sigma_{ZX}$ finite, p.d., non-singular by LLN.

c) $\sqrt{n}\left(\widehat{\beta}_{IV} - \beta\right) = \left(\left(\frac{1}{n}X'Z\right)A^{-1}\left(\frac{1}{n}Z'X\right)\right)^{-1}\left(\left(\frac{1}{n}X'Z\right)A^{-1}\left(\frac{1}{\sqrt{n}}Z'u\right)\right) \xrightarrow{d} N(0, G),$
where $G = \left\{(\Sigma_{ZX}' A^{-1}\Sigma_{ZX})^{-1}(\Sigma_{ZX}' A^{-1})\right\}\left\{\sigma_u^2 E[zz']\right\}\left\{(A^{-1}\Sigma_{ZX})(\Sigma_{ZX}' A^{-1}\Sigma_{ZX})^{-1}\right\},$
since $\left(\frac{1}{\sqrt{n}}Z'u\right) = \sqrt{n}\left(\frac{1}{n}Z'u - 0\right) \xrightarrow{d} N(0, \sigma_u^2 E[zz'])$ by CLT.