Econ203B Homework #6 Anton Cheremukhin

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Problem 1.

(a) $X \sim N(\mu, \sigma^2)$, $\frac{X-\mu}{\sigma} \sim N(0,1)$, $E[X|a<X<b] - \mu = \sigma E\left[\frac{X-\mu}{\sigma} | \frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right] =$

$$\sigma \frac{\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} z\varphi(z)dz}{\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} \varphi(z)dz} = \sigma \frac{\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}dz}{\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}dz} = \sigma \frac{-\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\Big|_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}}}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} = \sigma \frac{\varphi\left(\frac{a-\mu}{\sigma}\right) - \varphi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}$$

$E\left[\left(\frac{X-\mu}{\sigma}\right)^2 | \frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right] =$

$$\frac{\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} z^2\varphi(z)dz}{\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} \varphi(z)dz} = \frac{\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} \frac{z^2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}dz}{\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}dz} = \frac{-\left[\frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\Big|_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} - \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}dz\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} = 1 + \frac{\frac{a-\mu}{\sigma}\varphi\left(\frac{a-\mu}{\sigma}\right) - \frac{b-\mu}{\sigma}\varphi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}$$

$Var[X|a<X<b] = \sigma^2 Var\left[\frac{X-\mu}{\sigma} | \frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right] =$

$\sigma^2 \left\{E\left[\left(\frac{X-\mu}{\sigma}\right)^2 | \frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right] - E\left[\frac{X-\mu}{\sigma} | \frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right]^2\right\} =$

$$\sigma^2 \left\{1 + \frac{\frac{a-\mu}{\sigma}\varphi\left(\frac{a-\mu}{\sigma}\right) - \frac{b-\mu}{\sigma}\varphi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} - \left[\frac{\varphi\left(\frac{a-\mu}{\sigma}\right) - \varphi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2\right\}$$

Our case applies if we put $\mu = 0$.

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(b) $Y_i = \begin{cases} X_i\beta + \varepsilon_i, & a < X_i\beta + \varepsilon_i < b \\ 0 & \text{otherwise} \end{cases} \quad \varepsilon_i \sim N(0, \sigma^2)$

$$E[Y_i|X_i, a < X_i\beta + \varepsilon_i < b] = X_i\beta + \sigma \frac{\varphi\left(\frac{a-X_i\beta}{\sigma}\right) - \varphi\left(\frac{b-X_i\beta}{\sigma}\right)}{\Phi\left(\frac{b-X_i\beta}{\sigma}\right) - \Phi\left(\frac{a-X_i\beta}{\sigma}\right)}$$

$$Var[Y_i|X_i, a < X_i\beta + \varepsilon_i < b] = \sigma^2 \left\{1 + \frac{\frac{a-X_i\beta}{\sigma}\varphi\left(\frac{a-X_i\beta}{\sigma}\right) - \frac{b-X_i\beta}{\sigma}\varphi\left(\frac{b-X_i\beta}{\sigma}\right)}{\Phi\left(\frac{b-X_i\beta}{\sigma}\right) - \Phi\left(\frac{a-X_i\beta}{\sigma}\right)} - \left[\frac{\varphi\left(\frac{a-X_i\beta}{\sigma}\right) - \varphi\left(\frac{b-X_i\beta}{\sigma}\right)}{\Phi\left(\frac{b-X_i\beta}{\sigma}\right) - \Phi\left(\frac{a-X_i\beta}{\sigma}\right)}\right]^2\right\}$$

Method 1: MLE

$L(\beta, \sigma) = \prod_{i=1}^n \Pr(Y_i = 0|X_i)^{1-D_i} \prod_{i=1}^n [\Pr(Y_i \neq 0|X_i) f(Y_i|X_i, Y_i \neq 0)]^{D_i} =$

$= \prod_{i=1}^n \left[1 - \Phi\left(\frac{b-X_i\beta}{\sigma}\right) + \Phi\left(\frac{a-X_i\beta}{\sigma}\right)\right]^{1-D_i} \left[\frac{1}{\sigma}\varphi\left(\frac{Y_i - X_i\beta}{\sigma}\right)\right]^{D_i} \to \max_{\beta,\sigma}$

Method 2: NLS

$$\sum_{i=1}^n \left[Y_i - X_i\beta - \sigma \frac{\varphi\left(\frac{a-X_i\beta}{\sigma}\right) - \varphi\left(\frac{b-X_i\beta}{\sigma}\right)}{\Phi\left(\frac{b-X_i\beta}{\sigma}\right) - \Phi\left(\frac{a-X_i\beta}{\sigma}\right)}\right]^2 \to \min_{\beta,\sigma}$$

Method 3: WNLS

$$\sum_{i=1}^n \frac{\left[Y_i - X_i\beta - \sigma \frac{\varphi\left(\frac{a-X_i\beta}{\sigma}\right) - \varphi\left(\frac{b-X_i\beta}{\sigma}\right)}{\Phi\left(\frac{b-X_i\beta}{\sigma}\right) - \Phi\left(\frac{a-X_i\beta}{\sigma}\right)}\right]^2}{\sigma^2 \left\{1 + \frac{\frac{a-X_i\beta}{\sigma}\varphi\left(\frac{a-X_i\beta}{\sigma}\right) - \frac{b-X_i\beta}{\sigma}\varphi\left(\frac{b-X_i\beta}{\sigma}\right)}{\Phi\left(\frac{b-X_i\beta}{\sigma}\right) - \Phi\left(\frac{a-X_i\beta}{\sigma}\right)} - \left[\frac{\varphi\left(\frac{a-X_i\beta}{\sigma}\right) - \varphi\left(\frac{b-X_i\beta}{\sigma}\right)}{\Phi\left(\frac{b-X_i\beta}{\sigma}\right) - \Phi\left(\frac{a-X_i\beta}{\sigma}\right)}\right]^2\right\}} \to \min_{\beta,\sigma}$$

Method 4: Heckman Two-Step Procedure

1)Estimate Probit of D_i on X_i and get a first estimate γ_0 of β_0/σ_0.

2)Assuming that $\sigma_0 = 1$ compute $\lambda_i = \lambda(X_i) = \frac{\varphi\left(\frac{a-X_i\beta_0}{\sigma_0}\right) - \varphi\left(\frac{b-X_i\beta_0}{\sigma_0}\right)}{\Phi\left(\frac{b-X_i\beta_0}{\sigma_0}\right) - \Phi\left(\frac{a-X_i\beta_0}{\sigma_0}\right)}$

and estimate β_1 and σ_1 by OLS of Y_i on X_i and λ_i.

3)Using estimates recursively repeat step 2 untill convergence.