Econ203B Homework #5 Anton Cheremukhin

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Hayashi Problem 7.1

(a) $\Pr[a(w) \neq 1] = \Pr[f(y|x,\theta)/f(y|x,\theta_0) \neq 1] = \Pr[f(y|x,\theta) \neq f(y|x,\theta_0)] > 0$

(b) Since $\log[x]$ is concave, $\log[a(w)]$ is a concave function of a non-constant random variable. Hence by Jensen's inequality $E[\log[a(w)]] < \log[E[a(w)]]$

(c) By law of iterated expectations:

$E[a(w)] = E[E[a(w)|x]] = E\left[\int \frac{f(y|x,\theta)}{f(y|x,\theta_0)} f(y|x,\theta_0)\,dy\right] = E\left[\int f(y|x,\theta)\,dy\right] = E[1] = 1$

(d) Combine the previous results: $E[\log[f(y|x,\theta)]] - E[\log[f(y|x,\theta_0)]] = E\left[\log \frac{f(y|x,\theta)}{f(y|x,\theta_0)}\right] =$
$= E[\log[a(w)]] < \log[E[a(w)]] = \log[1] = 0$

Hayashi Problem 7.2

(a) $\int f(y|x,\theta)\,dy = 1, \forall\theta \Rightarrow E[s(w,\theta)] = \int s(w,\theta) f(y|x,\theta)\,dy = \int \frac{\partial \log f(y|x,\theta)}{\partial\theta} f(y|x,\theta)\,dy =$
$= \int \frac{\partial f(y|x,\theta)}{\partial\theta} dy = \frac{\partial}{\partial\theta}\int f(y|x,\theta)\,dy = 0, \forall\theta$

(b) $\int s(w,\theta) f(y|x,\theta)\,dy = 0, \forall\theta \Rightarrow E[H(w,\theta)] + E\left[s(w,\theta)\,s(w,\theta)'\right] =$
$= \int H(w,\theta) f(y|x,\theta)\,dy + \int s(w,\theta)\,s(w,\theta)' f(y|x,\theta)\,dy =$
$= \int \frac{\partial}{\partial\theta} s(w,\theta) f(y|x,\theta)\,dy + \int s(w,\theta) \frac{\partial}{\partial\theta} f(y|x,\theta)\,dy = \frac{\partial}{\partial\theta}\int s(w,\theta) f(y|x,\theta)\,dy = 0, \forall\theta$

Hayashi Problem 7.3

(a) $L_U = \Sigma \log f(y|x,\beta,\sigma^2) = c - \frac{n}{2}\log\sigma^2 - \frac{\Sigma(y-x\beta)^2}{2\sigma^2} \to \max_{\beta,\sigma^2}$

$\frac{\Sigma x'(y-x\beta)}{\sigma^2} = 0 \qquad \frac{n}{\sigma^2} = \frac{\Sigma(y-x\beta)^2}{(\sigma^2)^2} \qquad \Rightarrow \qquad \beta^U_{ML} = (X'X)^{-1}X'Y \qquad \sigma^2_U = \frac{1}{n}\Sigma(y - x\beta_U)^2$

$\frac{1}{2}(Y - X\beta)'(Y - X\beta) \to \min_\beta \qquad \Rightarrow -X'(Y - X\beta) = 0 \Rightarrow \beta^U_{OLS} = (X'X)^{-1}X'Y = \beta^U_{ML}$

$L_R = c - \frac{n}{2}\log\sigma^2 - \frac{\Sigma(y-x\beta)^2}{2\sigma^2} - \lambda(R\beta - c) \to \max_{\beta,\sigma^2}$

$\frac{x'(y-x\beta_R)}{\sigma^2_R} - R'\lambda' = 0 \qquad \frac{n}{\sigma^2_R} = \frac{\Sigma(y-x\beta_R)^2}{(\sigma^2_R)^2} \qquad R\beta_R = c \qquad \Rightarrow \qquad \sigma^2_R = \frac{1}{n}\Sigma(y - x\beta_R)^2$

$R'\lambda' 2\sigma^2_R = X'Y - (X'X)\beta_R \qquad \Rightarrow \qquad \beta_R = (X'X)^{-1}(X'Y - R'\lambda' 2\sigma^2_R)$

$R\beta_R = R\beta_U - \left[R(X'X)^{-1}R'\right]\lambda' 2\sigma^2 = c \qquad \Rightarrow \qquad \lambda' = \frac{1}{2\sigma^2}\left[R(X'X)^{-1}R'\right]^{-1}(R\beta_U - c)$

$\Rightarrow \qquad \beta^R_{ML} = \beta^U_{ML} - (X'X)^{-1}R'\left[R(X'X)^{-1}R'\right]^{-1}\left(R\beta^U_{ML} - c\right)$

$\frac{1}{2}(Y - X\beta)'(Y - X\beta) - \lambda(R\beta - c) \to \min_\beta$

$-X'(Y - X\beta) - R'\lambda' = 0 \qquad R\beta = c \qquad \Rightarrow \beta = (X'X)^{-1}(X'Y - R'\lambda') = \beta_0 - (X'X)^{-1}R'\lambda'$

$R\beta = R\beta_0 - R(X'X)^{-1}R'\lambda' \qquad \lambda' = \left[R(X'X)^{-1}R'\right]^{-1}(R\beta_0 - c)$

$\beta^R_{OLS} = \beta^U_{OLS} - (X'X)^{-1}R'\left[R(X'X)^{-1}R'\right]^{-1}\left(R\beta^U_{OLS} - c\right) = \beta^R_{ML}$

So, ML and OLS estimates coincide in both the restricted and unrestricted cases.

(b) From the previous analysis we have seen already that

$\sigma^2_U = \frac{1}{n}\Sigma(y - x\beta_U)^2 = \frac{1}{n}SSR_U \qquad \sigma^2_R = \frac{1}{n}\Sigma(y - x\beta_R)^2 = \frac{1}{n}SSR_R.$

So, $Q_n(\theta_i) = \frac{1}{n}\Sigma\log f(y|x,\beta,\sigma^2) = -\frac{\log(2\pi)}{2} - \frac{SSR_i}{2} - \frac{1}{2} \qquad i = U, R \qquad \theta = \{\beta, \sigma^2\}.$

(c) Theoretical ML: $M_i = c - \frac{1}{2}\log\sigma^2 - \frac{(y-x\beta)^2}{2\sigma^2} - \lambda(R\beta - c) \to \max_{\beta,\sigma^2}$

$\frac{\partial^2 M}{\partial\beta\partial\beta'} = -\frac{x'x}{\sigma^2} \qquad \frac{\partial^2 M}{\partial\beta\partial\sigma^2} = -\frac{x'(y-x\beta)}{(\sigma^2)^2} = 0 \qquad \frac{\partial^2 M}{\partial\sigma^2\partial\sigma^{2\prime}} = \frac{1}{2(\sigma^2)^2} - \frac{(y-x\beta)^2}{(\sigma^2)^3} \qquad i = U, R$

$I = E[-H(x,\theta)] = E\left[-\frac{\partial^2 M}{\partial\theta\partial\theta'}\right] = \begin{bmatrix} \frac{E[x'x]}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{bmatrix}$

Now let's find the second derivatives of our estimators:

$\frac{\partial^2 L_i}{\partial\beta\partial\beta'} = -\frac{\Sigma x'x}{2\sigma^2_i} \qquad \frac{\partial^2 L_i}{\partial\beta\partial\sigma^2} = -\frac{\Sigma x'(y-x\beta_i)}{2\left(\sigma^2_i\right)^2} = 0 \qquad \frac{\partial^2 L_i}{\partial\sigma^2\partial\sigma^{2\prime}} = \frac{n}{2\left(\sigma^2_i\right)^2} - \frac{\Sigma(y-x\beta_i)^2}{\left(\sigma^2_i\right)^3} \qquad i = U, R$

Sinse ML estimators are consistent, by continuity and Slutsky, Σ_i are consistent:

$$\underset{n\to\infty}{plim}\Sigma_i = \underset{n\to\infty}{plim}\left(-\frac{1}{n}\frac{\partial^2 L_i}{\partial\theta\partial\theta'}\right) = \begin{bmatrix} \frac{\underset{n\to\infty}{plim}\frac{1}{n}\Sigma x'x}{\sigma^2} & 0 \\ 0 & \frac{1}{2(\sigma^2)^2} - \frac{\underset{n\to\infty}{plim}\frac{1}{n}\Sigma(y-x\beta_i)^2}{(\sigma^2)^3} \end{bmatrix} = \begin{bmatrix} \frac{E[x'x]}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{bmatrix}$$

(d) $a(\theta) = R\beta - c \qquad A(\theta) = \frac{\partial a(\theta)}{\partial\theta} = \left[\frac{\partial[R\beta-c]}{\partial\beta}, \frac{\partial[R\beta-c]}{\partial\sigma^2}\right] = [R, 0]$

$W = na(\theta_U)'\left[A(\theta_U)(\Sigma_U)^{-1}A(\theta_U)'\right]^{-1}a(\theta_U) = n\left[R\beta_U - c\right]'\left[R\left(\frac{X'X}{SSR_U}\right)^{-1}R'\right]^{-1}\left[R\beta_U - c\right]$

$Q(\theta) = c - \frac{1}{2}\log\sigma^2 - \frac{\frac{1}{n}\Sigma(y-x\beta)^2}{2\sigma^2} \qquad \frac{\partial Q(\theta_R)}{\partial\beta} = \left[\frac{\frac{1}{n}\Sigma x'(y-x\beta)}{\sigma^2}, -\frac{1}{2\sigma^2} + \frac{\frac{1}{n}\Sigma(y-x\beta)^2}{2(\sigma^2)^2}\right] = \left[\frac{X'Y - X'X\beta}{SSR}, 0\right]$

$LM = n\left(\frac{\partial Q(\theta_R)}{\partial\beta}\right)'(\Sigma_R)^{-1}\left(\frac{\partial Q(\theta_R)}{\partial\beta}\right) = n\left(\frac{X'Y - X'X\beta_R}{SSR_R}\right)'\left(\frac{X'X}{SSR_R}\right)^{-1}\left(\frac{X'Y - X'X\beta_R}{SSR_R}\right) =$

$= n\left(Y - X\beta_R\right)'X\left(X'X\right)^{-1}X'\left(Y - X\beta_R\right)/SSR_R$

$LR = 2n\left[Q(\theta_U) - Q(\theta_R)\right] = n\left[\not{c} - \log\left(\frac{SSR_U}{n}\right) - \not{c} + \log\left(\frac{SSR_R}{n}\right)\right]$

(e) It immediately follows from:

$(Y - X\beta_U)'X(X'X)^{-1}X'(Y - X\beta_U) = \left[R\beta_U - c\right]'\left[R(X'X)^{-1}R'\right]^{-1}\left[R\beta_U - c\right] = SSR_R - SSR_U$

That $W = n\frac{SSR_R - SSR_U}{SSR_U} \qquad LM = n\frac{SSR_R - SSR_U}{SSR_R} \qquad LR = n\log\frac{SSR_R}{SSR_U}$

(f) It immediately follows from: $SSR_R > SSR_U$ that W>LR>LM since $x - 1 > \log x > 1 - \frac{1}{x}$.

Problem 1

Case 1: NLS. $\quad E[y|x] = \frac{\exp[x\beta]}{1+\exp[x\beta]} = G(x\beta) \in C^2 \qquad \beta_{NLS} = \arg\min\Sigma\left(y - G(x\beta)\right)^2$

Assumptions: $y = G(x\beta) + \varepsilon$, $\{x, \varepsilon\} - iid, E[\varepsilon|x] = 0, \frac{1}{n}\Sigma\left(y - G(x\beta)\right)^2 \overset{p}{\rightrightarrows} E\left[\left(y - G(x\beta)\right)^2\right]$

$\left[G(x\beta) = G(x\beta') \Rightarrow \beta = \beta'\right]$, β is in the interior of some compact set we look at.

Given that, $\exists\beta_{NLS} \overset{p}{\to} \beta$. $\qquad \frac{\partial G}{\partial\beta} = x'\frac{e^{x\beta}}{2e^{x\beta} + e^{2x\beta} + 1}$

$m(x, y, \beta) = \frac{1}{2}\left(y - G(x\beta)\right)'\left(y - G(x\beta)\right) \qquad s(x, y, \beta) = \frac{\partial m}{\partial\beta} = -\frac{\partial G(x\beta)}{\partial\beta}\left(y - G(x\beta)\right)$

$H(x, y, \beta) = \frac{\partial^2 m}{\partial\beta\partial\beta'} = \frac{\partial G(x\beta)}{\partial\beta}\frac{\partial G(x\beta)}{\partial\beta'} - \frac{\partial^2 G(x\beta)}{\partial\beta\partial\beta'}\left[y - G(x\beta)\right]$

$\Omega = ss' = \frac{\partial G(x\beta)}{\partial\beta}\left(y - G(x\beta)\right)\left(y - G(x\beta)\right)'\frac{\partial G(x\beta)}{\partial\beta}'$

More assumptions: $\exists E[\Omega] > 0, \exists E[H] > 0, E[\sup\|H\|] < \infty$

Given that, $\sqrt{n}\left(\beta_{NLS} - \beta\right) \overset{d}{\to} N\left(0, H^{-1}\Omega H^{-1}\right)$

Case 2: WNLS. $\quad \beta_{WNLS} = \arg\min\Sigma\frac{(y-G(x\beta))^2}{G(x\beta)[1-G(x\beta)]}$

Assumptions: $y = G(x\beta) + \varepsilon$, $\{x, \varepsilon\} - iid, E[\varepsilon|x] = 0, \frac{1}{n}\Sigma\frac{(y-G(x\beta))^2}{G(x\beta)[1-G(x\beta)]} \overset{p}{\rightrightarrows} E\left[\frac{(y-G(x\beta))^2}{G(x\beta)[1-G(x\beta)]}\right]$

$\left[G(x\beta) = G(x\beta') \Rightarrow \beta = \beta'\right]$, β is in the interior of some compact set we look at.

Given that, $\exists\beta_{NLS} \overset{p}{\to} \beta$.

$\frac{\partial G}{\partial\beta} = x'\frac{e^{x\beta}}{2e^{x\beta} + e^{2x\beta} + 1} \qquad m(x, y, \beta) = \frac{1}{2}\left(y - G(x,\beta)\right)'\left(y - G(x,\beta)\right)/\left[G(x,\beta)\left(1 - G(x,\beta)\right)\right]$

$s(x, y, \beta) = \frac{\partial m}{\partial\beta} = \frac{\partial G(x,\beta)}{\partial\beta}\left[-\frac{y - G(x,\beta)}{G(x,\beta)[1-G(x,\beta)]} - \frac{(1-2G(x\beta))(y-G(x\beta))^2}{2[G(x\beta)(1-G(x\beta))]^2}\right] \qquad \Omega = ss'$

$$H(x, y, \beta) = \frac{\partial^2 m}{\partial\beta\partial\beta'} = \frac{\partial G(x\beta)}{\partial\beta}\left[\begin{array}{c} \frac{1}{G(x,\beta)[1-G(x,\beta)]} + \frac{(1-2G(x\beta))(y-G(x,\beta))}{[G(x\beta)(1-G(x\beta))]^2} + \frac{(y-G(x\beta))^2}{[G(x\beta)(1-G(x\beta))]^2} \\ + \frac{(1-2G(x\beta))(y-G(x\beta))}{[G(x\beta)(1-G(x\beta))]^2} + \frac{(1-2G(x\beta))^2(y-G(x\beta))^2}{[G(x\beta)(1-G(x\beta))]^3} \end{array}\right]\frac{\partial G(x\beta)}{\partial\beta'}$$

$-\frac{\partial^2 G(x\beta)}{\partial\beta\partial\beta'}\left[-\frac{y - G(x,\beta)}{G(x,\beta)[1-G(x,\beta)]} - \frac{(1-2G(x\beta))(y-G(x\beta))^2}{2[G(x\beta)(1-G(x\beta))]^2}\right]$ (that's a mess of course, but that's what it is)

More assumptions: $\exists E[\Omega] > 0, \exists E[H] > 0, E[\sup\|H\|] < \infty$

Given that, $\sqrt{n}\left(\beta_{NLS} - \beta\right) \overset{d}{\to} N\left(0, H^{-1}\Omega H^{-1}\right)$

Problem 2

LPM is just standard OLS. In the other two cases the beta estimates could be done by NLS. Here is how we find standard errors for NLS (not WNLS).

Logit: $S = -\frac{\partial G(x\beta)}{\partial \beta}\left(y - G\left(x\beta\right)\right) = -x'\frac{e^{x\beta}}{\left(e^{x\beta}+1\right)^2}\left(y - \frac{e^{x\beta}}{e^{x\beta}+1}\right)$

$\Omega = ss' = x'\left[\frac{e^{x\beta}}{\left(e^{x\beta}+1\right)^2}\left(y - \frac{e^{x\beta}}{e^{x\beta}+1}\right)\right]\left[\frac{e^{x\beta}}{\left(e^{x\beta}+1\right)^2}\left(y - \frac{e^{x\beta}}{e^{x\beta}+1}\right)\right]' x$

$H = x'\left[\left(\frac{e^{x\beta}}{\left(e^{x\beta}+1\right)^2}\right)\left(\frac{e^{x\beta}}{\left(e^{x\beta}+1\right)^2}\right)' - \left(y - \frac{e^{x\beta}}{e^{x\beta}+1}\right)\left(\frac{e^{x\beta}\left(1-e^{x\beta}\right)}{\left(e^{x\beta}+1\right)^3}\right)'\right] x$

Probit:

$S = \frac{\partial}{\partial \beta}\left(y - \int_{-\infty}^{x\beta}\frac{\exp\left(-\frac{u^2}{2}\right)}{\sqrt{2\pi}}du\right)^2 = -x'\left(y - \int_{-\infty}^{x\beta}\frac{\exp\left(-\frac{u^2}{2}\right)}{\sqrt{2\pi}}du\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2\beta^2}$

$\Omega = ss' = x'\left[\left(y - \int_{-\infty}^{x\beta}\frac{\exp\left(-\frac{u^2}{2}\right)}{\sqrt{2\pi}}du\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2\beta^2}\right]\left[\left(y - \int_{-\infty}^{x\beta}\frac{\exp\left(-\frac{u^2}{2}\right)}{\sqrt{2\pi}}du\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2\beta^2}\right]' x$

$H = x'\left[\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2\beta^2}\right)\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2\beta^2}\right)' + \left(y - \int_{-\infty}^{x\beta}\frac{\exp\left(-\frac{u^2}{2}\right)}{\sqrt{2\pi}}du\right)\left(\frac{x\beta}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2\beta^2}\right)'\right] x$

Using these kxk matrices we compute the standard errors from $Avar\left(\beta\right) = H^{-1}\Omega H^{-1}$.

Marginal effects could be estimated using the formula: $\frac{\partial \Pr}{\partial x_{ik}} = f\left(\bar{x}\beta\right)\beta_k$.

Using the result from problem 1: $\sqrt{n}\left(\beta_{NLS} - \beta\right) \xrightarrow{d} N\left(0, H^{-1}\Omega H^{-1}\right)$ and applying delta-method for the function: $u\left(\beta\right) = f\left(\bar{x}\beta\right)\beta \Rightarrow \quad \Psi = \frac{\partial u(\beta)}{\partial \beta} = \frac{\partial f(\bar{x}\beta)}{\partial \beta}\beta + f\left(\bar{x}\beta\right)$ we get:

$$\sqrt{n}\left(u\left(\beta_{NLS}\right) - u\left(\beta\right)\right) \xrightarrow{d} N\left(0, \Psi' H^{-1}\Omega H^{-1}\Psi\right)$$

Probit: $F\left(a\right) = \int_{-\infty}^{a}\frac{\exp\left(-\frac{u^2}{2}\right)}{\sqrt{2\pi}}du \quad \Rightarrow \quad f\left(a\right) = \frac{e^{-\frac{1}{2}a^2}}{\sqrt{2\pi}} \Rightarrow$

$\frac{\partial f(a)}{\partial a} = -\frac{a}{\sqrt{2\pi}}e^{-\frac{1}{2}a^2} \Rightarrow \Psi = \frac{e^{-\frac{1}{2}(\bar{x}\beta)^2}}{\sqrt{2\pi}}\left[\left(\bar{x}\beta\right)^2 + 1\right]$

Logit: $F\left(a\right) = \frac{e^a}{e^a+1} \quad f\left(a\right) = \frac{e^a}{\left(e^a+1\right)^2} \quad \frac{\partial f(a)}{\partial a} = \frac{e^a\left(1-e^a\right)}{\left(e^a+1\right)^3} \quad \Psi = \frac{e^{\bar{x}\beta}\left(1-e^{\bar{x}\beta}\right)}{\left(e^{\bar{x}\beta}+1\right)^3}\left(\bar{x}\beta\right) + \frac{e^{\bar{x}\beta}}{\left(e^{\bar{x}\beta}+1\right)^2}$

Results (Logit already rescaled):

LPM	c*	KL6*	K618	WA*	WE*	UN	CIT	PRIN*	LWW*
beta	0.69	-0.291	-0.008	-0.012	0.042	-0.004	-0.005	-6.8e-6	0.09
st.er.	0.16	0.037	0.014	0.003	0.009	0.0055	0.037	1.5e-6	0.03

Probit	c	KL6*	K618	WA*	WE*	UN	CIT	PRIN*	LWW*
beta	0.62	-0.93	-0.033	-0.037	0.131	-0.012	-0.01	-2.2e-5	0.28
st.er.	0.51	0.15	0.048	0.008	0.031	0.017	0.11	0.6e-5	0.11

Logit	c	KL6*	K618	WA*	WE*	UN	CIT	PRIN*	LWW*
beta	0.39	-0.82	-0.015	-0.032	0.122	-0.006	-0.01	-2.1e-6	0.28
st.er.	0.48	0.17	0.047	0.008	0.031	0.015	0.10	0.6e-6	0.11

Marginal Effects of LWW:

Model	LPM	Probit	Logit	Logit Rescaled
marg.effect	0.093275	0.11072	0.28475	0.14988
st.er.	0.031982	0.04639	0.04959	0.050941

All the models have approximately the same fit according to the graph and the tables. The marginal effects are slightly different, though of the same magnitude and significance.

To Compare Fit of the Models Plot CDFs of Resiguals: