Econ203B Homework #2 Anton Cheremukhin

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Exercise 1

The power of a test is the probability to reject a false null hypothesis. The alternative hypothesis here is $H_1 : \beta = \beta$. So we can compute the probability of H_0 being rejected given that H_1 is true. $\hat{\beta} \sim N\left(\beta, \sigma_\beta^2\right), Z_0 = \hat{\beta}/\sigma_\beta \sim N(\beta, 1), U = Z_0 - \beta = \hat{\beta}/\sigma_\beta - \beta \sim N(0,1)$

Power of the test is equal to $\Pr(|Z_0| > 1.645|\beta) = \Pr(U > 1.645 - \beta \cup U < -1.645 - \beta) = \Phi(-1.645 - \beta) + 1 - \Phi(1.645 - \beta)$. The result does not depend on σ_β^2.

-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
0.6388	0.4432	0.2635	0.1421	0.1	0.1421	0.2635	0.4432	0.6388

Exercise 2

(a) Try to compute the probability of H_0 being rejected given that H_1 is true: $\hat{\beta} \sim N(\beta, \Sigma)$,

$W_0 = \left(\hat{\beta} - \beta_0\right)' \Sigma^{-1} \left(\hat{\beta} - \beta_0\right) = \hat{\beta}'\Sigma^{-1}\hat{\beta} = \frac{\hat{\beta}_1^2 + \hat{\beta}_2^2 - 2r\hat{\beta}_1\hat{\beta}_2}{1-r^2} \sim \chi^2(2)$ under H_0

$W_1 = \left(\hat{\beta} - \beta\right)' \Sigma^{-1} \left(\hat{\beta} - \beta\right) = W_0 + \frac{\beta_1^2 + \beta_2^2 - 2r\beta_1\beta_2}{1-r^2} - 2\frac{\hat{\beta}_1\beta_1 + \hat{\beta}_2\beta_2}{1+r} \sim \chi^2(2)$ under H_1

Here the chi-square and the normal $\hat{\beta}$ are not independent so we can't compute the result analytically. We shall estimate the power numerically by applying the Monte-Carlo technique with 10^5 iterations. To do so we generate realizations of $\hat{\beta} \sim N(\beta, \Sigma)$ given each of the 9 true pairs of β. For that we decompose $CC' = \Sigma$ and generate $\hat{\beta} = \beta + Cz$ where $z = N(0, I)$.

$$\Sigma = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{1-r^2} & r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{1-r^2} & 0 \\ r & 1 \end{bmatrix} = CC'$$

Then calculate W_0 and the probability of H_0 being rejected in each case. Power of the test is equal to $\Pr(W_0 > 5.99|\beta)$.

r=0.6	$(-1,1)$	$(-1,0)$	$(-1,-1)$	$(0,1)$	$(0,0)$	$(0,-1)$	$(1,1)$	$(1,0)$	$(1,-1)$
	0.50	0.185	0.155	0.185	0.05	0.185	0.155	0.185	0.50
r=-0.6	$(-1,1)$	$(-1,0)$	$(-1,-1)$	$(0,1)$	$(0,0)$	$(0,-1)$	$(1,1)$	$(1,0)$	$(1,-1)$
	0.155	0.185	0.50	0.185	0.05	0.185	0.50	0.185	0.155

The results rotated 90 degrees, so that powers for anticorrelated betas and correlated betas swapped. The absolute values did not change as suggested by our analytical analisis.

Exercise 3

(a) $\Pr(Y_i = y|\, X_i = x) = e^{-\beta x}(\beta x)^y / y!$ $\quad \log\Pr(Y_i = y|\, X_i = x) = -x\beta + y\log(x\beta) - \log y!$

$L = \log \Pi_{i=1}^n \Pr(Y_i = y|\, X_i = x) = \Sigma_{i=1}^n \log\Pr(Y_i = y|\, X_i = x) = \Sigma_{i=1}^n (-x_i\beta + y_i \log(x_i\beta) - \log y_i!)$

FOC: $\frac{\partial}{\partial\beta}\Sigma_{i=1}^n(-x_i\beta + y_i\log(x_i\beta) - \log y_i!) = \Sigma_{i=1}^n\left(-x_i + \frac{y_i}{\beta}\right) = -n\left(\bar{x} - \frac{\bar{y}}{\beta}\right) = 0 \Rightarrow \hat{\beta}_{ML} = \bar{y}/\bar{x}$

(b) By LLN $\bar{y} \xrightarrow{p} E[Y], \bar{x} \xrightarrow{p} E[X]$ by continuity $\frac{1}{\bar{x}} \xrightarrow{p} [E[X]]^{-1}$ by Slutsky $\hat{\beta}_{ML} \xrightarrow{p} E[Y]/E[X]$

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(c) $x_i \sim iid$, $y_i \sim iid, Poisson(x_i)$ $\quad n\bar{y} = \Sigma Poisson(x_i) = Poisson(\Sigma x_i) = Poisson(n\bar{x})$

$\hat{\beta}_{ML} = \bar{y}/\bar{x} \sim \frac{1}{n} Poisson(n\bar{x}) \to \frac{1}{n} Poisson(nE[X])$

Exercise 4

$$Y = X\beta + \varepsilon, \qquad \varepsilon|X \sim N\left(0, \beta^2 I_n\right), \qquad Y|X \sim N\left(X\beta, \beta^2 I_n\right)$$

$$L = \log \Pi \frac{1}{\sqrt{2\pi\beta^2}} \exp\left(-\frac{(y_i - x_i\beta)^2}{2\beta^2}\right) = c + \Sigma\left(-\log\beta - \frac{\left(\frac{y_i}{\beta} - x_i\right)^2}{2}\right) \to \max_\beta$$

$$\text{FOC:} \qquad \Sigma\left(-\frac{1}{\beta} + \left(\frac{y_i}{\beta} - x_i\right)\frac{y_i}{\beta^2}\right) = -\frac{n}{\beta^3}\left(\beta^2 + \overline{x_i y_i}\beta - \overline{y_i^2}\right) = 0$$

$$\hat{\beta}_{ML} = -\frac{1}{2n}\left(X'Y \pm \sqrt{(X'Y)^2 - 4nY'Y}\right)$$

$$\text{SOC:} \qquad \Sigma\left(\frac{1}{\beta^2} + \frac{2}{\beta^3}x_i y_i - \frac{3}{\beta^4}y_i^2\right) = \frac{n}{\beta^4}\left(\beta^2 + 2\overline{x_i y_i}\beta - 3\overline{y_i^2}\right) < 0$$

$$\hat{\beta}_{ML} \in \left[-\frac{1}{n}\left(X'Y + \sqrt{(X'Y)^2 - 3nY'Y}\right); -\frac{1}{n}\left(X'Y - \sqrt{(X'Y)^2 - 3nY'Y}\right)\right]$$

$$I(\beta|X) = -E\left[\Sigma\left(\frac{1}{\beta^2} + \frac{2}{\beta^3}x_i y_i - \frac{3}{\beta^4}y_i^2\right)\middle| X\right] = -E\left[\Sigma \frac{1}{\beta^2} + \frac{2}{\beta^3}x_i(x_i\beta + \varepsilon) - \frac{3}{\beta^4}(x_i\beta + \varepsilon)^2\middle| X\right] =$$

$$-\Sigma E\left[\frac{1 - x_i^2}{\beta^2} + x_i\left(1 - \frac{6}{\beta^3}\right)\varepsilon - \frac{3}{\beta^4}\varepsilon^2\middle| X\right] = -\Sigma\left[\frac{1 - x_i^2}{\beta^2} + x_i\left(1 - \frac{6}{\beta^3}\right)E[\varepsilon|X] - \frac{3}{\beta^4}E[\varepsilon^2|X]\right] = \Sigma\frac{x_i^2 + 2}{\beta^2}$$

$I(\beta|X)^{-1} = \frac{\beta^2}{X'X + 2n}$ – the Cramer–Rao Lower Bound on the variance of $\hat{\beta}_{ML}$.

Exercise 5

$$Y_i = X_i\beta + \varepsilon_i, \qquad \varepsilon_i|X_i \sim N\left(0, (Z_i\gamma)^2\right), \qquad Y_i|X_i \sim N\left(X_i\beta, (Z_i\gamma)^2\right)$$

$$L = \log \Pi \frac{1}{\sqrt{2\pi(z_i\gamma)^2}} \exp\left(-\frac{(y_i - x_i\beta)'(y_i - x_i\beta)}{2(z_i\gamma)^2}\right) = c + \Sigma\left(-\log(z_i\gamma) - \frac{(y_i - x_i\beta)'(y_i - x_i\beta)}{2(z_i\gamma)^2}\right) \to \max_{(\beta,\gamma)}$$

$$\text{FOC:} \qquad \frac{\partial L}{\partial \beta} = \Sigma\frac{x_i'(y_i - x_i\beta)}{(z_i\gamma)^2} = 0 \qquad \frac{\partial L}{\partial \gamma} = \Sigma z_i'\left(-\frac{1}{z_i\gamma} + \frac{(y_i - x_i\beta)'(y_i - x_i\beta)}{(z_i\gamma)^3}\right) = 0$$

$$\text{SOC:} \quad \frac{\partial^2 L}{\partial\beta\partial\beta'} = -\Sigma\frac{x_i x_i'}{(z_i\gamma)^2} \quad \frac{\partial^2 L}{\partial\beta\partial\gamma'} = -2\Sigma\frac{x_i z_i'(y_i - x_i\beta)}{(z_i\gamma)^3} \quad \frac{\partial^2 L}{\partial\gamma\partial\gamma'} = \Sigma z_i z_i'\left(\frac{1}{(z_i\gamma)^2} - 3\frac{(y_i - x_i\beta)'(y_i - x_i\beta)}{(z_i\gamma)^4}\right)$$

$$I(\beta, \gamma|X, Z) = -E\left[\begin{array}{cc} -\Sigma\frac{x_i x_i'}{(z_i\gamma)^2} & -2\Sigma\frac{(y_i - x_i\beta)}{(z_i\gamma)^3}x_i z_i' \\ -2\Sigma\frac{(y_i - x_i\beta)}{(z_i\gamma)^3}z_i x_i' & \Sigma z_i z_i'\left(\frac{1}{(z_i\gamma)^2} - 3\frac{(y_i - x_i\beta)'(y_i - x_i\beta)}{(z_i\gamma)^4}\right)\end{array}\middle| X, Z\right] =$$

$$E\left[\begin{array}{cc} \Sigma\frac{x_i x_i'}{(z_i\gamma)^2} & 2\Sigma\frac{x_i z_i'\varepsilon_i}{(z_i\gamma)^3} \\ 2\Sigma\frac{z_i x_i'\varepsilon_i}{(z_i\gamma)^3} & \Sigma z_i z_i'\left(3\frac{\varepsilon_i{}'\varepsilon_i}{(z_i\gamma)^4} - \frac{1}{(z_i\gamma)^2}\right)\end{array}\middle| X, Z\right] = \left[\begin{array}{cc} \Sigma\frac{x_i x_i'}{(z_i\gamma)^2} & 0 \\ 0 & 2\Sigma\frac{z_i z_i'}{(z_i\gamma)^2}\end{array}\right]$$

Exercise 6

$$\hat{\beta}_{OLS} = (X'X)^{-1}X'Y = \left[\begin{array}{c} \frac{2}{3} \\ \frac{11}{6}\end{array}\right] = \left[\begin{array}{c} 0.67 \\ 1.83\end{array}\right] \qquad s^2 = \frac{e'e}{n-k} = \frac{55}{18}$$

$$\widehat{Var}\left[\hat{\beta}_{OLS}\right] = \left[\begin{array}{cc} \frac{121}{54} & -\frac{11}{27} \\ -\frac{11}{27} & \frac{11}{108}\end{array}\right] \qquad s.e.\left[\hat{\beta}_{OLS}\right] = \left[\begin{array}{c} 1.50 \\ 0.32\end{array}\right]$$

$$\hat{\beta}_{GLS} = (X'V^{-1}X)^{-1}X'V^{-1}Y = \left[\begin{array}{c} \frac{2471}{2028} \\ \frac{26\,759}{16\,224}\end{array}\right] = \left[\begin{array}{c} 1.22 \\ 1.65\end{array}\right] \qquad s^2 = \frac{e'e}{n-k} = \frac{35\,875}{194\,688}$$

$$\widehat{Var}\left[\hat{\beta}_{OLS}\right] = \left[\begin{array}{cc} \frac{4484\,375}{12\,338\,352} & -\frac{14\,888\,125}{98\,706\,816} \\ -\frac{14\,888\,125}{98\,706\,816} & \frac{78\,530\,375}{789\,654\,528}\end{array}\right] \qquad s.e.\left[\hat{\beta}_{OLS}\right] = \left[\begin{array}{c} 0.603 \\ 0.315\end{array}\right]$$

Exercise 7

$$E[Y|X] = X\beta \qquad \hat{\beta}_{OLS} = (X'X)^{-1}X'Y$$

$$Y^* = TY \qquad X^* = TX \qquad E[Y^*|X^*] = E[TY|TX] = TX\beta = X^*\beta$$

$$\hat{\beta}_{GLS} = (X^{*\prime}X^*)^{-1}X^{*\prime}Y^* = (X'T'TX)^{-1}X'T'TY$$

Only if $T'T = I$, is it that $(X'T'TX)^{-1}X'T'TY = (X'X)^{-1}X'Y$ and the estimates coincide.