Economics 203A Homework # 6 Anton Cheremukhin

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Exercise 1 *Let Y_n be a statistic such that $\lim_{n\to\infty} E[Y_n] = \theta$ and $\lim_{n\to\infty} \sigma^2_{Y_n} = 0$. Prove that Y_n is a consistent estimator of θ. Hint: $E\left[(Y_n-\theta)^2\right] = (E[Y_n-\theta])^2 + \sigma^2_{Y_n}$. Why?*

Proof. $\Pr[|Y_n-\theta| \geq \varepsilon] \leq \frac{E\left[(Y_n-\theta)^2\right]}{\varepsilon^2} = \frac{E\left[(Y_n-E[Y_n]+E[Y_n]-\theta)^2\right]}{\varepsilon^2} =$
$\frac{E\left[(Y_n-E[Y_n])^2\right]}{\varepsilon^2} + \frac{(E[Y_n]-\theta)^2}{\varepsilon^2} + \frac{2E[Y_n-E[Y_n]](E[Y_n]-\theta)}{\varepsilon^2} = \frac{\sigma^2_{Y_n}+(E[Y_n-\theta])^2}{\varepsilon^2}$
$\lim_{n\to\infty} \Pr[|Y_n-\theta| \geq \varepsilon] \leq \lim_{n\to\infty} \frac{\sigma^2_{Y_n}+(E[Y_n-\theta])^2}{\varepsilon^2} = 0 \qquad \Rightarrow \qquad p\lim_{n\to\infty} Y_n = \theta$
i.e Y_n is a consistent estimator of θ. ■

Exercise 2 *Let $X_1,\ldots,X_n$ represent a random sample from the pdf $f(x) = (1/\theta)\exp(-x/\theta)$, $0 < x < \infty$, zero elsewhere. Find the MLE $\widehat{\theta}$ of θ. Also, find the MLE of $\Pr(X \leq 2)$*

Proof. $L(x_i,\theta) = \prod_i f(x_i,\theta) = \prod_{i=1}^n \frac{1}{\theta}\exp\left(-\frac{x_i}{\theta}\right) = \theta^{-n}\exp\left(-\frac{\sum_{i=1}^n x_i}{\theta}\right) \to \max_\theta$
FOC: $-n\theta^{-1-n}\exp\left(-\frac{\sum_{i=1}^n x_i}{\theta}\right) + \theta^{-n}\exp\left(-\frac{\sum_{i=1}^n x_i}{\theta}\right)\frac{\sum_{i=1}^n x_i}{\theta^2} = 0$
$\hat{\theta} = \frac{1}{n}\sum_{i=1}^n x_i = \bar{x}$
$\Pr(X \leq 2) = \int_0^2 \exp\left(-\frac{x}{\theta}\right) d\frac{x}{\theta} = \int_0^{2\theta} \exp(-x)\, dx = 1 - e^{-2\theta}$
MLE of $\Pr(X \leq 2)$ is equal to $1 - e^{-2\hat{\theta}}$ due to invariance principle. ■

Exercise 3 *Let $X' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 5 & 2 \end{bmatrix}$, $y' = \begin{bmatrix} 14 & 17 & 8 & 16 & 3 \end{bmatrix}$,*
Calculate the following:

(a) $X'X$
(b) $\det(X'X)$
(c) $(X'X)^{-1}$
(d) $(X'X)^{-1}X'$
(e) $(X'X)^{-1}X'y$
(f) $X(X'X)^{-1}X'$
(g) $\operatorname{trace}\left(X(X'X)^{-1}X'\right)$
(h) $I - X(X'X)^{-1}X'$
(i) $\operatorname{trace}\left(I - X(X'X)^{-1}X'\right)$
(j) $\left(I - X(X'X)^{-1}X'\right)y$

Proof. $X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 3 \\ 1 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 16 \\ 16 & 58 \end{bmatrix}$

$\det(X'X) = \det\begin{bmatrix} 5 & 16 \\ 16 & 58 \end{bmatrix} = 34$

$(X'X)^{-1} = \begin{bmatrix} 5 & 16 \\ 16 & 58 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{29}{17} & -\frac{8}{17} \\ -\frac{8}{17} & \frac{5}{34} \end{bmatrix}$

$(X'X)^{-1}X' = \begin{bmatrix} \frac{29}{17} & -\frac{8}{17} \\ -\frac{8}{17} & \frac{5}{34} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \frac{13}{17} & -\frac{3}{17} & \frac{5}{17} & -\frac{11}{17} & \frac{13}{17} \\ -\frac{3}{17} & \frac{2}{17} & -\frac{1}{34} & \frac{9}{34} & -\frac{3}{17} \end{bmatrix}$

$$
(X'X)^{-1} X'y = \begin{bmatrix} \frac{13}{17} & -\frac{3}{17} & \frac{5}{17} & -\frac{11}{17} & \frac{13}{17} \\ -\frac{3}{17} & \frac{2}{17} & -\frac{1}{34} & \frac{9}{34} & -\frac{3}{17} \end{bmatrix} \begin{bmatrix} 14 \\ 17 \\ 8 \\ 16 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}
$$

$$
X (X'X)^{-1} X' = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 3 \\ 1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{13}{17} & -\frac{3}{17} & \frac{5}{17} & -\frac{11}{17} & \frac{13}{17} \\ -\frac{3}{17} & \frac{2}{17} & -\frac{1}{34} & \frac{9}{34} & -\frac{3}{17} \end{bmatrix} = \begin{bmatrix} \frac{7}{17} & \frac{1}{17} & \frac{4}{17} & -\frac{2}{17} & \frac{7}{17} \\ \frac{1}{17} & \frac{5}{17} & \frac{3}{17} & \frac{7}{17} & \frac{1}{17} \\ \frac{4}{17} & \frac{3}{17} & \frac{7}{34} & \frac{5}{34} & \frac{4}{17} \\ -\frac{2}{17} & \frac{7}{17} & \frac{5}{34} & \frac{23}{34} & -\frac{2}{17} \\ \frac{7}{17} & \frac{1}{17} & \frac{4}{17} & -\frac{2}{17} & \frac{7}{17} \end{bmatrix}
$$

$$
\text{trace}\left(X (X'X)^{-1} X'\right) = 2
$$

$$
I - X (X'X)^{-1} X' = I - \begin{bmatrix} \frac{7}{17} & \frac{1}{17} & \frac{4}{17} & -\frac{2}{17} & \frac{7}{17} \\ \frac{1}{17} & \frac{5}{17} & \frac{3}{17} & \frac{7}{17} & \frac{1}{17} \\ \frac{4}{17} & \frac{3}{17} & \frac{7}{34} & \frac{5}{34} & \frac{4}{17} \\ -\frac{2}{17} & \frac{7}{17} & \frac{5}{34} & \frac{23}{34} & -\frac{2}{17} \\ \frac{7}{17} & \frac{1}{17} & \frac{4}{17} & -\frac{2}{17} & \frac{7}{17} \end{bmatrix} = \begin{bmatrix} \frac{10}{17} & -\frac{1}{17} & -\frac{4}{17} & \frac{2}{17} & -\frac{7}{17} \\ -\frac{1}{17} & \frac{12}{17} & -\frac{3}{17} & -\frac{7}{17} & -\frac{1}{17} \\ -\frac{4}{17} & -\frac{3}{17} & \frac{27}{34} & -\frac{5}{34} & -\frac{4}{17} \\ \frac{2}{17} & -\frac{7}{17} & -\frac{5}{34} & \frac{11}{34} & \frac{2}{17} \\ -\frac{7}{17} & -\frac{1}{17} & -\frac{4}{17} & \frac{2}{17} & \frac{10}{17} \end{bmatrix}
$$

$$
\text{trace}\left(I - X (X'X)^{-1} X'\right) = -1
$$

$$
\left(I - X (X'X)^{-1} X'\right) y = \begin{bmatrix} \frac{10}{17} & -\frac{1}{17} & -\frac{4}{17} & \frac{2}{17} & -\frac{7}{17} \\ -\frac{1}{17} & \frac{12}{17} & -\frac{3}{17} & -\frac{7}{17} & -\frac{1}{17} \\ -\frac{4}{17} & -\frac{3}{17} & \frac{27}{34} & -\frac{5}{34} & -\frac{4}{17} \\ \frac{2}{17} & -\frac{7}{17} & -\frac{5}{34} & \frac{11}{34} & \frac{2}{17} \\ -\frac{7}{17} & -\frac{1}{17} & -\frac{4}{17} & \frac{2}{17} & \frac{10}{17} \end{bmatrix} \begin{bmatrix} 14 \\ 17 \\ 8 \\ 16 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -3 \\ -1 \\ -5 \end{bmatrix} \quad \blacksquare
$$

Exercise 4 *Consider the classical regression model* $y = \underset{n\times k}{X} \beta + \varepsilon$ *where* $\varepsilon \sim N(0, \sigma^2 I)$
(a) Let $P = X (X'X)^{-1} X'$. *Show that* $P = P'$ *and* $PP = P$
(b) Let $M = I - P$. *Show that* $M' = M$, $MM = M$, *and* $\text{trace}(M) = n - k$
(c) Let $e = y - X\widehat{\beta}$, *where* $\widehat{\beta} = (X'X)^{-1} X'y$. *Show that* $e = M\varepsilon$
(d) Show that $\widehat{\beta} = \beta + (X'X)^{-1} X'\varepsilon$.
(e) Show that $\widehat{\beta} \sim N\left(\beta, \sigma^2 (X'X)^{-1}\right)$
(f) Show that $\widehat{\beta}$ *is independent of* e
(g) Let $s^2 = e'e/(n-k)$. *Show that* $\left(\widehat{\beta} - \beta\right) \Big/ \sqrt{s^2 (X'X)^{-1}} \sim t(n-k)$.
(h) Assume that $k = 1$. *Show that the interval* $\left(\widehat{\beta} - 1.96\sqrt{\sigma^2 (X'X)^{-1}}, \widehat{\beta} + 1.96\sqrt{\sigma^2 (X'X)^{-1}}\right)$ *contains* β *with 95% probability. Show that the interval* $\left(\widehat{\beta} - 2.576\sqrt{\sigma^2 (X'X)^{-1}}, \widehat{\beta} + 2.576\sqrt{\sigma^2 (X'X)^{-1}}\right)$ *contains* β *with 99% probability.*
(i) Assume that $n = 30$ *and* $k = 1$. *Also assume that* σ^2 *is unknown. Finally assume that* $s^2 = 1$ *and* $X'X = 25$. *Construct a 95% confidence interval for* β.

Proof. $P' = (X (X'X)^{-1} X')' = (X')'((X'X)^{-1})'X' = X (X'(X')')^{-1} X' = P$
$PP = X (X'X)^{-1} X'X (X'X)^{-1} X' = X (X'X)^{-1} X' = P$
$M' = (I - P)' = I - P' = I - P = M$
$MM = (I - P)(I - P) = I - 2P + PP = I - P = M$
$\text{trace}(M) = \text{trace}(I - P) = \text{trace}(I) - \text{trace}(P) = n - k$

$MX = (I - P)X = (X - X(X'X)^{-1}X'X) = X - X = 0$
$e = y - X(X'X)^{-1}X'y = (I-P)y = My = M(X\beta + \varepsilon) = M\varepsilon$
$\widehat{\beta} = (X'X)^{-1}X'y = (X'X)^{-1}X'(X\beta+\varepsilon) = \beta + (X'X)^{-1}X'\varepsilon$
$\varepsilon \sim N(0, \sigma^2 I) \qquad \Rightarrow \qquad \widehat{\beta} \sim N\left(\beta, \sigma^2 (X'X)^{-1} X'X (X'X)^{-1}\right) = N\left(\beta, \sigma^2 (X'X)^{-1}\right)$
$e = M\varepsilon \sim N(0, \sigma^2 MM') = N(0, \sigma^2 MM) = N(0, \sigma^2 M)$
Hence, e and $\widehat{\beta}$ have joint normal distribution.
$E[(X'X)^{-1}X'\varepsilon \cdot (M\varepsilon)'] = \sigma^2 (X'X)^{-1} X'M' = \sigma^2 (X'X)^{-1}(MX)' = 0$
Hence, they are independent.
$s^2/\sigma^2 = e'e/(\sigma^2(n-k)) \sim \chi^2(n-k)$ is independent of $\widehat{\beta}$.
$\left(\widehat{\beta} - \beta\right) \Big/ \sqrt{\sigma^2 (X'X)^{-1}} \sim N(0, I)$
Hence, $\left(\widehat{\beta} - \beta\right) \Big/ \sqrt{s^2 (X'X)^{-1}} \sim t(n-k)$
$\Pr\left[\widehat{\beta} - 1.96\sqrt{\sigma^2 (X'X)^{-1}} < \beta < \widehat{\beta} + 1.96\sqrt{\sigma^2 (X'X)^{-1}}\right] =$
$\Pr\left[\left|\left(\widehat{\beta} - \beta\right) \Big/ \sqrt{\sigma^2 (X'X)^{-1}}\right| < 1.96\right] = 0.95$
$\Pr\left[\widehat{\beta} - 2.576\sqrt{\sigma^2 (X'X)^{-1}} < \beta < \widehat{\beta} + 2.576\sqrt{\sigma^2 (X'X)^{-1}}\right] =$
$\Pr\left[\left|\left(\widehat{\beta} - \beta\right) \Big/ \sqrt{\sigma^2 (X'X)^{-1}}\right| < 2.576\right] = 0.99$
$\left(\widehat{\beta} - \beta\right) \Big/ \sqrt{1/25} = 5\left(\widehat{\beta} - \beta\right) \sim t(n-k)$
$t_{0.05}(29) = 2.045231 \qquad \Rightarrow$
$\Pr\left[\widehat{\beta} - 0.409 < \beta < \widehat{\beta} + 0.409\right] = 0.95$ ∎

Exercise 5 *Consider the classical regression model* $y = \underset{n\times k}{X}\beta + \varepsilon$ *where* $E[\varepsilon] = 0$ *and* $E[\varepsilon\varepsilon'] = \sigma^2 I_n$. *Notice that we are not assuming that* ε *has a normal distribution here.*
(a) Show that $\widehat{\beta} = (X'X)^{-1}X'y$ *is an unbiased estimator for* β.
(b) Show that $s^2 = e'e/(n-k)$ *is an unbiased estimator for* σ^2.

Proof. $\widehat{\beta} = \beta + (X'X)^{-1}X'\varepsilon$
$E\left[\widehat{\beta}\right] = E[\beta] + E\left[(X'X)^{-1}X'\varepsilon\right] = \beta + (X'X)^{-1}X'E[\varepsilon] = \beta$
$e = M\varepsilon \qquad e'e = \varepsilon' M' M \varepsilon = \varepsilon' MM\varepsilon = \varepsilon' M \varepsilon$ (see previous exercise)
$\varepsilon' M\varepsilon = \text{trace}(\varepsilon' M \varepsilon)$ because it is an 1x1 scalar.
$\text{trace}(\varepsilon' M\varepsilon) = \text{trace}(M\varepsilon\varepsilon')$ by the properties of trace
$E[e'e] = E[\varepsilon' M \varepsilon] = E[\text{trace}(\varepsilon' M\varepsilon)] = E[\text{trace}(M\varepsilon\varepsilon')] = \text{trace}(ME[\varepsilon\varepsilon']) =$
$\text{trace}(MI_n) = \sigma^2\,\text{trace}(M) = \sigma^2(n-k)$
Hence $E[s^2] = E[e'e/(n-k)] = \sigma^2 \Rightarrow s^2$ is an unbiased estimator of σ^2. ∎

Exercise 6 *Suppose that* $U_1, \ldots, U_n$ *are i.i.d.* $N(\mu, \sigma^2)$. *Show that* $\left(\overline{U} - \mu\right) \Big/ \left(\frac{\sigma}{\sqrt{n}}\right) \sim N(0,1)$, *based on which show that* $\Pr\left[\overline{U} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \overline{U} + 1.96\frac{\sigma}{\sqrt{n}}\right] = 95\%$ *and*

$$\Pr\left[\overline{U} - 2.576\frac{\sigma}{\sqrt{n}} < \mu < \overline{U} + 2.576\frac{\sigma}{\sqrt{n}}\right] = 99\%$$

Proof. $U_i \sim iid\ N(\mu, \sigma^2) \quad \Rightarrow \quad (U_i - \mu)/\sigma \sim iid\ (0,1) \quad \Rightarrow$
$(\overline{U} - \mu)/\sigma \sim N(0, 1/n) \quad \Rightarrow \quad \sqrt{n}\left(\overline{U} - \mu\right)/\sigma \sim N(0,1)$
Hence, $\Pr\left[\overline{U} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \overline{U} + 1.96\frac{\sigma}{\sqrt{n}}\right] = \Pr\left[\left|\sqrt{n}\left(\overline{U} - \mu\right)/\sigma\right| < 1.96\right] = 95\%$
$\Pr\left[\overline{U} - 2.576\frac{\sigma}{\sqrt{n}} < \mu < \overline{U} + 2.576\frac{\sigma}{\sqrt{n}}\right] = \Pr\left[\left|\sqrt{n}\left(\overline{U} - \mu\right)/\sigma\right| < 2.576\right] = 99\%$ ∎

Exercise 7 *Let the observed value of the sample mean* $\overline{X}$ *of a random sample of size 20 from* $N(\mu, 80)$ *be 81.2. Find a 95% confidence interval for* μ

Proof. $CI(\mu) = (81.2 \pm 1.96\frac{\sqrt{80}}{\sqrt{20}}) = (77.28,\ 85.12)$ ∎

Exercise 8 *Let* $\overline{X}$ *be the sample mean of a random sample of size* n *from* $N(\mu, 9)$*. Find* n *such that* $\Pr\left(\overline{X} - 1 < \mu < \overline{X} + 1\right) = .90$ *approximately.*

Proof. $1 = 1.645\frac{3}{\sqrt{n}} \quad \Rightarrow \quad n = (1.645 * 3)^2 \approx 24$ ∎

Exercise 9 *Let a random sample of size 17 from* $N(\mu, \sigma^2)$ *yield* $\overline{x} = 4.7$ *and* $s^2 = 6.12$*. Determine a 95% confidence interval for* μ*.*

Proof. $CI(\mu) = (4.7 \pm 2.12\sqrt{6.12/17}) = (3.428,\ 5.972)$ ∎

Exercise 10 *Let two independent random samples, each of size 10, from two normal distributions* $N(\mu_1, 1)$ *and* $N(\mu_2, 1)$ *yield* $\overline{x}_1 = 4.8$, $\overline{x}_2 = 5.6$*. Find a 95% confidence interval for* $\mu_1 - \mu_2$*.*

Proof. $CI(\mu_1 - \mu_2) = ((4.8 - 5.6) \pm 1.96\sqrt{0.1 + 0.1}) = (-1.6765,\ 0.0765)$ ∎

Exercise 11 *Let* $\overline{X}_n$ *denote the mean of a random sample of size* n *from a gamma distribution with parameters* $\alpha = \mu > 0$ *and* $\beta = 1$*. Show that the limiting distribution of* $\sqrt{n}\left(\overline{X}_n - \mu\right)/\sqrt{\overline{X}_n}$ *is* $N(0,1)$

Proof. $E[X_i] = \mu = Var[X_i]$
by CLT $\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} N(0, \mu)$
by LLN $\overline{X}_n \xrightarrow{p} \mu$
bu Slutsky $\sqrt{n}\left(\overline{X}_n - \mu\right)/\sqrt{\overline{X}_n} \xrightarrow{d} N(0,1)$ ∎

Exercise 12 *Let* $\overline{X}_n$ *and* s_n^2 *represent, respectively, the mean and variance of a random sample of size* n *from* $N(\mu, \sigma^2)$*. Prove that the limiting distribution of* $\sqrt{n}\left(\overline{X}_n - \mu\right)/s_n$ *is* $N(0,1)$*.*

Proof. by CLT $\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$
by LLN $s_n^2 \xrightarrow{p} \sigma^2$
bu Slutsky $\sqrt{n}\left(\overline{X}_n - \mu\right)/\sqrt{s_n^2} \xrightarrow{d} N(0,1)$ ∎

Exercise 13 *Let* $\overline{x}$ *be the observed mean of a random sample of size* n *from a distribution having mean* μ *and known variance* σ^2*. Find* n *so that* $\left(\overline{x} - \frac{\sigma}{4}, \overline{x} + \frac{\sigma}{4}\right)$ *is an approximate 95% confidence interval for* μ*.*

Proof. by CLT $\sqrt{n}(\overline{x} - \mu) \xrightarrow{d} N(0, \sigma^2)$
Hence, $\Pr[|\sqrt{n}(\overline{x} - \mu)|/\sigma \le 1.96] = 0.95$
That means $\Pr[\overline{x} - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{x} + 1.96\frac{\sigma}{\sqrt{n}}] = 0.95$
$\frac{1}{4} \approx \frac{1.96}{\sqrt{n}} \quad \Rightarrow \quad n \approx (1.96 * 4)^2 \approx 62$ ∎