Economics 203A Homework # 5 Anton Cheremukhin

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Exercise 1 *Let $Y_n \sim b(n,p)$. Prove that Y_n/n converges in probability to p.*

Proof. Using Chebyshev inequality: $\Pr\left[\left|\frac{Y_n}{n} - E\left[\frac{Y_n}{n}\right]\right| \geq \varepsilon\right] \leq \frac{Var\left[\frac{Y_n}{n}\right]}{\varepsilon^2} \Leftrightarrow$
$\Leftrightarrow \Pr\left[\left|\frac{Y_n}{n} - p\right| \geq \varepsilon\right] \leq \frac{p(1-p)n}{n^2\varepsilon^2} \quad \Rightarrow \quad \text{plim}_{n\to\infty} \frac{Y_n}{n} = p.$ ■

Exercise 2 *Let W_n denote a random variable with mean μ and variance b/n^p with $p > 0$. Prove that W_n converges in probability to μ.*

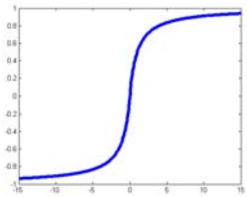
Proof. $\Pr[|W_n - \mu| \geq \varepsilon] \leq \frac{Var[W_n]}{\varepsilon^2} = \frac{b}{n^p\varepsilon^2} \Rightarrow \text{plim}_{n\to\infty} W_n = \mu.$ ■

Exercise 3 *Suppose that X_1 and X_2 are* i.i.d. *$N(0,1)$ random variables. Let*

$$Z \equiv \max\left(\frac{X_1}{1+|X_1|}, \frac{X_2}{1+|X_2|}\right).$$

Using 1000 rounds of Monte Carlo simulation, compute $\Pr[Z \leq .5]$.

Proof. The function is bounded: $Z \leq 1$, so $\Pr[Z \leq 5] = 1$.



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Exercise 4 *(Importance Sampling) Suppose that your computer does not generate a random variable with p.d.f. $f(\cdot)$, but it does generate one with p.d.f. $g(\cdot)$. Assume that $f(x) = 0$ whenever $g(x) = 0$. Assume that X is a random variable with p.d.f. equal to $g(x)$.*

1. *Let $F(\cdot)$ denote the c.d.f. of the random variable with p.d.f. equal to $f(\cdot)$. Show that*

$$F(t) = E\left[I(X \leq t)\frac{f(X)}{g(X)}\right]$$

Here, $I(A)$ is an indicator function which equals 1 if A is satisfied, and 0 otherwise. For example, $I(X \leq t) = 1$ if $X \leq t$, and $I(X \leq t) = 0$ if $X > t$.

2. *Suppose that you are given an* i.i.d. *sequence $\{X_i\}$ whose p.d.f. equals $g(\cdot)$. Show that*

$$\underset{n\to\infty}{\text{plim}} \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq t)\frac{f(X_i)}{g(X_i)} = F(t).$$

3. Suppose that the p.d.f. of a random variable Y equals $f(\cdot)$. Show that

$$E[\phi(Y)] = E\left[\phi(X)\frac{f(X)}{g(X)}\right],$$

where X is the random variable with p.d.f. equal to $g(\cdot)$.

4. Show that

$$\operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \phi(X_i)\frac{f(X_i)}{g(X_i)} = E[\phi(Y)],$$

where $\{X_i\}$ is the i.i.d. *sequence of random variables with p.d.f. equal to $g(\cdot)$.*

Proof. $X \sim g(\cdot), Y \sim f(\cdot) \qquad F(t) = \Pr[Y \le t] = E[I(Y \le t)] =$
$\int_{f(y)>0} I(y \le t) f(y)dy = \int_{g(x)>0} I(x \le t)\frac{f(x)}{g(x)}g(x)dx = E\left[I(X \le t)\frac{f(X)}{g(X)}\right].$
By law of large numbers $\operatorname{plim}_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} I(X_i \le t)\frac{f(X_i)}{g(X_i)} = E\left[I(X \le t)\frac{f(X)}{g(X)}\right] = F(t)$.
$E[\phi(Y)] = \int_{f(y)>0}\phi(y) f(y)dy = \int_{g(x)>0}\phi(x)\frac{f(x)}{g(x)}g(x)dx = E\left[\phi(X)\frac{f(X)}{g(X)}\right].$
Again by law of large numbers $\operatorname{plim}_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\phi(X_i)\frac{f(X_i)}{g(X_i)} = E\left[\phi(X)\frac{f(X)}{g(X)}\right] = E[\phi(Y)]$. ■

Exercise 5 *Let X have the pdf $f(x) = x^2/9$, $0 < x < 3$, zero elsewhere. Find the pdf of $Y = X^3$*

Proof. $y = u(x) = x^3 \Rightarrow x = u^{-1}(y) = \sqrt[3]{y}$ - a one-to-one transformation.
$g(y) = f(u^{-1}(y))\left|\frac{du^{-1}(y)}{dy}\right| = (\sqrt[3]{y}^2/9)\left|\frac{1}{3}\frac{\sqrt[3]{y}}{y}\right| = \frac{1}{27}$, when $0 < y < 27$, zero elsewhere. ■

Exercise 6 *Let X_1 and X_2 have the joint pdf $h(x_1, x_2) = 2\exp(-x_1 - x_2)$, $0 < x_1 < x_2 < \infty$, zero elsewhere. Find the joint pdf of $Y_1 = 2X_1$ and $Y_2 = X_2 - X_1$, and prove that they are independent.*

Proof. $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Y = u(X) = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X = u^{-1}(Y) = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
$g(y) = h(u^{-1}(y))\left|\frac{du^{-1}(y)}{dy}\right| = 2\exp\left(-(\frac{1}{2}y_1) - (\frac{1}{2}y_1 + y_2)\right)\frac{1}{2} =$
$g(y) = \exp(-y_1 - y_2)$, $0 < y_1 < \infty, 0 < y_2 < \infty$ ■

Exercise 7 *Assume that X is a Cauchy random variable: its p.d.f. at x is equal to*

$$\frac{1}{\pi(1 + x^2)}$$

We want to compute

$$E\left[\frac{\exp(X)}{1 + \exp(X)}\right]$$

Generate 1,000 i.i.d. *Cauchy random variables $X_1, \ldots, X_{1000}$. Compute*

$$\frac{1}{1000}\sum_{i=1}^{n}\frac{\exp(X_i)}{1 + \exp(X_i)}.$$

Why is it a sensible approximation?

Proof. The Cauchy distributed X can be generated as a ratio of two standard normal variables. The answer is always in the vicinity of 0.5. Because X is centered around zero, half of the time X is positive and half of the fime negative. At the same time both parts have infinite mean. Hence the variable $\frac{\exp(X)}{1+\exp(X)}$behaves similar to the Bernulli variable which is 0 or 1 with equal probabilities. That's why the mean is 1/2. ■

Exercise 8 *Find the mean and variance of the beta distribution.*

Proof. $E[Y] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y^\alpha (1-y)^{\beta-1} dy = \frac{\Gamma(\alpha+1)/\Gamma(\alpha)}{\Gamma(\alpha+\beta+1)/\Gamma(\alpha+\beta)} = \frac{\alpha}{\alpha+\beta}$
$E[Y^2] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y^{\alpha+1} (1-y)^{\beta-1} dy = \frac{\Gamma(\alpha+2)/\Gamma(\alpha)}{\Gamma(\alpha+\beta+2)/\Gamma(\alpha+\beta)} = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$
$Var[Y] = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha}{\alpha+\beta}\frac{\alpha}{\alpha+\beta} = \frac{(\alpha^2+\alpha)(\alpha+\beta)-\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ ■

Exercise 9 *Let X_1 and X_2 be independent random variables. Let $X_1 \sim \chi^2(r_1)$ and $Y = X_1 + X_2 \sim \chi^2(r)$. Show that $X_2 \sim \chi^2(r - r_1)$.*

Proof. $E\left[e^{tY}\right] = (1-2t)^{-r/2} \qquad E\left[e^{tX_1}\right] = (1-2t)^{-r_1/2}$
$E\left[e^{tY}\right] = E\left[e^{tX_1}\right] E\left[e^{tX_2}\right] = (1-2t)^{-r_1/2} E\left[e^{tX_2}\right] = (1-2t)^{-r/2}$
$E\left[e^{tX_1}\right] = (1-2t)^{-(r-r_1)/2} \qquad \Rightarrow \qquad X_2 \sim \chi^2(r-r_1)$ ■

Exercise 10 *Let $X_1, \ldots, X_n$ denote mutually independent random variables with moment generating functions $M_1(t), \ldots, M_n(t)$. Show that the moment generating function of $Y = \sum_{i=1}^n k_i X_i$ is $\prod_{i=1}^{n} M_i(k_i t)$. Use this result to show that, if $X_1, \ldots, X_n$ are independent Poisson random variables with means μ_i, then $\sum_{i=1}^n X_i$ is Poisson with mean $\sum_{i=1}^n \mu_i$*

Proof. $M(t) = E\left[e^{tY}\right] = E\left[e^{t\sum_{i=1}^n k_i X_i}\right] = E\left[\prod_{i=1}^{n} e^{tk_i X_i}\right] = \prod_{i=1}^{n} E\left[e^{(tk_i)X_i}\right] = \prod_{i=1}^{n} M_i(k_i t)$
$X_1, \ldots, X_n$ are iid Poisson, $\qquad Y = \sum_{i=1}^n X_i$
Hence, $M(t) = \prod_{i=1}^{n} M_i(t) = \prod_{i=1}^{n} e^{\mu_i(e^t-1)} = e^{(e^t-1)\sum_{i=1}^n \mu_i}$ - Poisson with $\mu = \sum_{i=1}^n \mu_i$. ■