Economics 203a Homework # 3 Anton Cheremukhin

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Exercise 1 *If the moment generating function of a random variable X is* $\left(\frac{1}{3}+\frac{2}{3}e^t\right)^5$, *find* $\Pr(X=2 \text{ or } 3)$

Proof. $(1-p+pe^t)^n = \sum_y \binom{n}{y} p^y (1-p)^{n-y}$, *hence* $n=5, p=2/3$

$\Pr(X=2) = \binom{5}{2}\left(\frac{2}{3}\right)^2\left(1-\frac{2}{3}\right)^{5-2} = \frac{40}{243}$

$\Pr(X=3) = \binom{5}{2}\left(\frac{2}{3}\right)^3\left(1-\frac{2}{3}\right)^{5-3} = \frac{80}{243}$

$\Pr(X=3 \text{ or } 2) = \Pr(X=3)+\Pr(X=2) = \frac{120}{243}$ ■

Exercise 2 *If* $X \sim b(n,p)$ *show that* $E\left(\frac{X}{n}\right)=p, \quad E\left[\left(\frac{X}{n}-p\right)^2\right] = \frac{p(1-p)}{n}$

Proof. $E(X/n) = E(X)/n = np/n = p$

$E\left[\left(\frac{X}{n}-p\right)^2\right] = E\left[(X-pn)^2\right]/n^2 = \frac{p(1-p)n}{n^2} = \frac{p(1-p)}{n}$ ■

Exercise 3 *Compute the skewness and kurtosis of the binomial distribution* $b(n,p)$

Proof. $E\left[(X-\mu)^3\right] = E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3$

$E\left[(X-\mu)^4\right] = E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 4\mu^3 E[X] + \mu^4$

$E[X] = \frac{\partial}{\partial t}((1-p+pe^t)^n)\big|_{t=0} = np$

$E[X^2] = \frac{\partial^2}{\partial t^2}((1-p+pe^t)^n)\Big|_{t=0} = np - np^2 + n^2p^2$

$E[X^3] = \frac{\partial^3}{\partial t^3}((1-p+pe^t)^n)\Big|_{t=0} = np - 3np^2 + 2np^3 + 3n^2p^2 - 3n^2p^3 + n^3p^3$

$E[X^4] = \frac{\partial^4}{\partial t^4}((1-p+pe^t)^n)\Big|_{t=0} =$

$np - 7np^2 + 12np^3 - 6np^4 + 7n^2p^2 - 18n^2p^3 + 11n^2p^4 + 6n^3p^3 - 6n^3p^4 + n^4p^4$

$E\left[(X-\mu)^3\right]/\sigma^3 = \frac{1}{n^2p^2(1-p)^3}[1 - 3p(1-n+n^2) + p^2(2-3n+4n^2-3n^3+3n^4-n^5)]$

$E\left[(X-\mu)^4\right]/\sigma^4 = \frac{1}{n^3p^3(1-p)^4}[1 + p(-7+7n-4n^2)p + p^2(2-3n+3n^2-2n^3+n^4) + +p^3(-6+11n-14n^2+13n^3-10n^4+6n^5-4n^6+n^7)]$ ■

Exercise 4 *If the random variable X has a Poisson distribution such that* $\Pr(X=1)=\Pr(X=2)$, *find* $\Pr(X=4)$

Proof. $\frac{m^1e^{-m}}{1!} = \frac{m^2e^{-m}}{2!} \Rightarrow m=2 \Rightarrow \Pr[X=4] = \frac{2^4e^{-2}}{4!} = \frac{2}{3e^2}$ ■

Exercise 5 *Let X has a Poisson distribution with mean equal to 100. Use Chebyshev's inequality to determine a lower bound for* $\Pr(75<X<125)$

Proof. $\Pr(75<X<125) = 1 - \Pr\left[|X-100| \geq \frac{25}{10}10\right] \geq 1 - \frac{100}{25^2} = \frac{21}{25}$ ■

Exercise 6 *Compute the skewness and kurtosis of the Poisson distribution with mean* μ.

Proof. $E[X^2] = \frac{\partial^2}{\partial t^2}(\exp[m(e^t-1)])\Big|_{t=0} = m + m^2$

$E[X^3] = \frac{\partial^3}{\partial t^3}(\exp[m(e^t-1)])\Big|_{t=0} = m + 3m^2 + m^3$

$E[X^4] = \frac{\partial^4}{\partial t^4}(\exp[m(e^t-1)])\Big|_{t=0} = m + 7m^2 + 6m^3 + m^4$

$E\left[(X-\mu)^3\right]/\sigma^3 = \frac{1}{m^3}(m + 3m^2 + m^3 - 3m^2 - 3m^3 + 3m^3 - m^3) = \frac{1}{m^2}$

$E\left[(X-\mu)^4\right]/\sigma^4 = \frac{1}{m^4}(m + 7m^2 + 6m^3 + m^4 - 4m^2 - 12m^3 - 4m^4 + 6m^3 + 6m^4 - 4m^4 + m^4) = \frac{1+3m}{m^3}$ ■