■

Exercise 18 *Let X be a random variable with mean μ and let $E\left[(X-\mu)^{2k}\right]$ exist. Show, with $d>0$, that $\Pr(|X-\mu|\geq d)\leq E\left[(X-\mu)^{2k}\right]\Big/ d^{2k}$*

Proof. *From Markov's inequality $\Pr[|Y|\geq c]\leq \frac{E[|Y|^m]}{c^m}$. Denote $Y=X-\mu$. Hence $\Pr(|X-\mu|\geq d)\leq E[(X-\mu)^m]/d^m$. Now let $m=2k$.* ■

Exercise 19 *If X is a random variable such that $E[X]=3$ and $E[X^2]=13$, use Chebyshev's inequality to determine a lower bound for the probability $\Pr(-2<X<8)$*

Proof. $\Pr(-2<X<8)=\Pr(|X-3|<5)=1-\Pr(|X-3|\geq 5)\geq 1-Var(X)/25=1-(E[X^2]-E[X]^2)/25=1-4/25=\frac{21}{25}$ ■

Exercise 20 *Let the random variables X and Y have the joint pdf*
(a) $f(x,y)=\frac{1}{3}$, $(x,y)=(0,0),(1,1)\,(2,2)$, zero elsewhere
(b) $f(x,y)=\frac{1}{3}$, $(x,y)=(0,2),(1,1)\,(2,0)$, zero elsewhere
(c) $f(x,y)=\frac{1}{3}$, $(x,y)=(0,0),(1,1)\,(2,1)$, zero elsewhere
In each case, compute the correlation coefficient of X and Y

Proof. *(a)* $Cov(X,Y)=E[XY]-E[X]E[Y]=(0+1+4)/3-1=\frac{2}{3}$, $Var(X)=E[XX]-E[X]E[X]=(0+1+4)/3-1=\frac{2}{3}$, $Var(Y)=E[YY]-E[Y]E[Y]=(0+1+4)/3-1=\frac{2}{3}$ $Corr[X,Y]=Cov(X,Y)/\sqrt{Var[X]Var[Y]}=1$

(b) $Cov(X,Y)=E[XY]-E[X]E[Y]=(0+1+0)/3-1=-\frac{2}{3}$, $Var(X)=E[XX]-E[X]E[X]=(0+1+4)/3-1=\frac{2}{3}$, $Var(Y)=E[YY]-E[Y]E[Y]=(0+1+4)/3-1=\frac{2}{3}$, $Corr[X,Y]=Cov(X,Y)/\sqrt{Var[X]Var[Y]}=-1$.

(c) $Cov(X,Y)=E[XY]-E[X]E[Y]=(0+1+2)/3-2/3=\frac{1}{3}$, $Var(X)=E[XX]-E[X]E[X]=(0+1+4)/3-1=\frac{2}{3}$, $Var(Y)=E[YY]-E[Y]E[Y]=(0+1+1)/3-(2/3)^2=\frac{2}{9}$, $Corr[X,Y]=Cov(X,Y)/\sqrt{Var[X]Var[Y]}=(1/3)/\sqrt{(2/3)*(2/9)}=\frac{1}{2}\sqrt{3}$. ■

Exercise 21 *If the,correlation coefficient ρ of X and Y exists, show that $-1\leq\rho\leq 1$. Hint: Consider the discriminant of the nonnegative quadratic function $h(t)=E\left\{[(X-\mu_X)+t(Y-\mu_Y)]^2\right\}$*

Proof. $h(t)=E\left\{[(X-\mu_X)+t(Y-\mu_Y)]^2\right\}=E[(X-\mu_X)^2]+2tE[(X-\mu_X)(Y-\mu_Y)]+t^2E[(Y-\mu_Y)^2]=Var[X]+2tCov[X,Y]+t^2Var[Y]$

$D/4=Cov[X,Y]^2-Var[X]*Var[Y]\leq 0$ *as expectation of a nonnegative number is non-negative for any t. Hence* $|Corr[X,Y]|=|Cov(X,Y)/\sqrt{Var[X]Var[Y]}|\leq 1$ ■

Exercise 22 *Show that the random variables X_1 and X_2 with joint pdf $f(x_1,x_2)=12x_1x_2(1-x_2)$, $0<x_1<1$, $0<x_2<1$, zero elsewhere, are independent.*

Proof. $f(x_1)=\int_0^1 12x_1x_2(1-x_2)\,dx_2=2x_1$

$f(x_2)=\int_0^1 12x_1x_2(1-x_2)\,dx_1=6x_2(1-x_2)$

i.e. $f(x_1)f(x_2)=f(x_1,x_2)$ *which means x_1 and x_2 are independent* ■

Exercise 23 *Let X and Y have the joint pdf $f(x,y)=3x$, $0<y<x<1$, zero elsewhere. Are X and Y independent?*

Proof. $f(y)=\int_y^1 3x\,dx=\frac{3}{2}(1-y^2)$ $\qquad f(x)=\int_0^x 3x\,dy=3x^2$

$f(y)f(x)=\frac{3}{2}(1-y^2)*3x^2=\frac{9}{2}x^2(1-y^2)\neq f(x,y)$ *which means X and Y are not independent.*

■