Exercise 18  Let $X$ be a random variable with mean $\mu$ and let $E \left[ (X - \mu)^{2k} \right]$ exist. Show, with $d > 0$, that $\Pr (|X - \mu| \geq d) \leq E \left[ (X - \mu)^{2k} \right] / d^{2k}$.

Proof. From Markov’s inequality $\Pr ([Y] \geq c) \leq \frac{E(|Y|^m)}{c^m}$. Denote $Y = X - \mu$. Hence $\Pr (|X - \mu| \geq d) \leq E \left[ (X - \mu)^m \right] / d^m$. Now let $m = 2k$.

Exercise 19  If $X$ is a random variable such that $E \left[ X \right] = 3$ and $E \left[ X^2 \right] = 13$, use Chebyshev’s inequality to determine a lower bound for the probability $\Pr (-2 < X < 8)$.

Proof. $\Pr (-2 < X < 8) = \Pr (|X - 3| < 5) \geq 1 - \Var(X)/25 = 1 - (E \left[ X^2 \right] - E \left[ X \right]^2)/25 = 1 - 4/25 = \frac{21}{25}$.

Exercise 20  Let the random variables $X$ and $Y$ have the joint pdf

(a) $f(x, y) = \frac{1}{2}$, $(x, y) = (0, 0), (1, 1), (2, 2)$, zero elsewhere
(b) $f(x, y) = \frac{1}{2}$, $(x, y) = (0, 2), (1, 1), (2, 0)$, zero elsewhere
(c) $f(x, y) = \frac{1}{3}$, $(x, y) = (0, 0), (1, 1), (2, 1)$, zero elsewhere

In each case, compute the correlation coefficient of $X$ and $Y$.

Proof. (a) $\Corr(X, Y) = E[XY] - E[X]E[Y] = (0 + 1 + 4)/3 - 1 = \frac{2}{3}$, $\Var(X) = E[X^2] - E[X]^2 = (0 + 1 + 2)/3 - 1 = \frac{2}{3}$, $\Var(Y) = E[Y^2] - E[Y]^2 = (0 + 1 + 4)/3 - 1 = \frac{2}{3}$, $\Corr(X, Y) = Cov(X, Y)/\sqrt{\Var(X)\Var(Y)} = 1$.

(b) $\Corr(X, Y) = E[XY] - E[X]E[Y] = (0+1+2)/3-1 = -\frac{2}{3}$, $\Var(X) = E[X^2] - E[X]^2 = (0 + 1 + 4)/3 - 1 = \frac{2}{3}$, $\Var(Y) = E[Y^2] - E[Y]^2 = (0 + 1 + 1)/3 - (2/3)^2 = \frac{2}{9}$, $\Corr(X, Y) = Cov(X, Y)/\sqrt{\Var(X)\Var(Y)} = -1$.

(c) $\Corr(X, Y) = E[XY] - E[X]E[Y] = (0+1+2)/3-2/3 = \frac{1}{3}$, $\Var(X) = E[X^2] - E[X]^2 = (0 + 1 + 4)/3 - 1 = \frac{2}{3}$, $\Var(Y) = E[Y^2] - E[Y]^2 = (0 + 1 + 1)/3 - (2/3)^2 = \frac{2}{9}$, $\Corr(X, Y) = Cov(X, Y)/\sqrt{\Var(X)\Var(Y)} = (1/3)/\sqrt{(2/3)*(2/9)} = \frac{1}{2}\sqrt{3}$.

Exercise 21  If the correlation coefficient $\rho$ of $X$ and $Y$ exists, show that $-1 \leq \rho \leq 1$. Hint: Consider the discriminant of the nonnegative quadratic function $h(t) = E \left[ [(X - \mu_X) + t(Y - \mu_Y)]^2 \right]$

Proof. $h(t) = E \left[ [(X - \mu_X) + t(Y - \mu_Y)]^2 \right] = E[(X - \mu_X)^2] + 2tE[(X - \mu_X)(Y - \mu_Y)] + t^2E[(Y - \mu_Y)^2] = Var[X] + 2tCov[X, Y] + t^2Var[Y]$

$D/4 = Cov[X, Y]^2 - Var[X]Var[Y] \leq 0$ as expectation of a nonnegative number is non-negative for any $t$. Hence $|\Corr(X, Y)| = |\Corr(X, Y)/\sqrt{\Var(X)\Var(Y)}| \leq 1$.

Exercise 22  Show that the random variables $X_1$ and $X_2$ with joint pdf $f(x_1, x_2) = 12x_1x_2(1 - x_2), 0 < x_1 < 1, 0 < x_2 < 1$, zero elsewhere, are independent.

Proof. $f(x_1) = \int_0^1 12x_1x_2(1 - x_2) \, dx_2 = 2x_1$

$f(x_2) = \int_0^1 12x_1x_2(1 - x_2) \, dx_1 = 6x_2(1 - x_2)$

i.e. $f(x_1)f(x_2) = f(x_1, x_2)$ which means $x_1$ and $x_2$ are independent.

Exercise 23  Let $X$ and $Y$ have the joint pdf $f(x, y) = 3x, 0 < x < y < 1$, zero elsewhere. Are $X$ and $Y$ independent?

Proof. $f(y) = \int_0^y 3xdx = \frac{3}{2}(1 - y^2)$

$f(x) = \int_0^x 3ydy = 3x^2$

$f(y)f(x) = \frac{3}{2}(1 - y^2) \cdot 3x^2 = \frac{9}{2}x^2(1 - y^2) \neq f(x, y)$ which means $X$ and $Y$ are not independent.