Economics 203A, Homework #1 Anton Cheremukhin

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Exercise 1 *If the sample space is* $\Omega = C_1 \cup C_2$ *and if* $P(C_1) = 0.8$ *and* $P(C_2) = 0.5$*, find* $P(C_1 \cap C_2)$.

Proof. $P(C_1 \cap C_2) = P(C_1) + P(C_2) - P(C_1 \cup C_2) = 0.8 + 0.5 - 1 = 0.3$ ■

Exercise 2 *If* $P(C) > 0$ *and if* $C_1, C_2, \ldots$ *are mutually disjoint sets, show that*

$$P(C_1 \cup C_2 \cup \cdots | C) = P(C_1 | C) + P(C_2 | C) + \cdots$$

Proof. $P(C_1 \cup C_2 \cup \cdots | C) = \frac{P((C_1 \cup C_2 \cup \cdots) \cap C)}{P(C)} = \frac{P((C_1 \cap C) \cup (C_2 \cap C) \cup \cdots)}{P(C)} = \frac{P(C_1 \cap C) + P(C_2 \cap C) + \cdots}{P(C)} = P(C_1 | C) + P(C_2 | C) + \cdots$ ■

Exercise 3 *Prove that*

$$P(C_1 \cap C_2 \cap C_3 \cap C_4) = P(C_1) P(C_2 | C_1) P(C_3 | C_1 \cap C_2) P(C_4 | C_1 \cap C_2 \cap C_3)$$

Proof. $P(C_1 \cap C_2 \cap C_3 \cap C_4) = P(C_1 \cap (C_2 \cap C_3 \cap C_4)) = P(C_1) P(C_2 \cap C_3 \cap C_4 | C_1) = P(C_1) P(C_2 | C_1) P(C_3 \cap C_4 | C_1 \cap C_2) = P(C_1) P(C_2 | C_1) P(C_3 | C_1 \cap C_2) P(C_4 | C_1 \cap C_2 \cap C_3)$ ■

Exercise 4 *Bowl 1 contains 3 red chips and 7 blue chips. Bowl 2 contains 6 red chips and 4 blue chips. A bowl is selected at random and then 1 chip is drawn from this bowl.*
(a) Compute the probability that this chip is red.
(b) Relative to the hypothesis that the chip is red, find the conditional probability that it is drawn from Bowl 2

Proof. *Define A={1,2} - number of bowl, B={r,g} - color of chip.*

(a) $P(B = r) = P(B = r \cap (A = 1 \cup A = 2)) = P(B = r \cap A = 1) + P(B = r \cap A = 2) = P(B = r | A = 1) P(A = 1) + P(B = r | A = 2) P(A = 2) = 3/10 * 1/2 + 6/10 * 1/2 = 9/20.$

(b) $P(A = 2 | B = r) = \frac{P(A=2 \cap B=r)}{P(B=r)} = \frac{P(B=r|A=2)P(A=2)}{P(B=r)} = \frac{6/20}{9/20} = 2/3$ ■

Exercise 5 *If* C_1 *and* C_2 *are independent events, show that* C_1 *and* C_2^c *are also independent.*

Proof. *If* C_1 *and* C_2 *are independent events, then* $P(C_2 | C_1) = P(C_2)$*, then* $P(C_2^c | C_1) = P(C_1 \cap C_2^c) / P(C_1) = (P(C_1) - P(C_1 \cap C_2)) / P(C_1) = 1 - P(C_2 | C_1) = 1 - P(C_2) = P(C_2^c)$ *which means that* C_1 *and* C_2^c *are independent events.* ■

Exercise 6 *Let* C_1 *and* C_2 *be independent events such that* $P(C_1) = 0.6$ *and* $P(C_2) = 0.3$*. Compute* $P(C_1 \cap C_2)$ *and* $P(C_1 \cup C_2)$

Proof. $P(C_1 \cap C_2) = P(C_2 | C_1) \cdot P(C_1) = P(C_2) \cdot P(C_1) = 0.18$

$P(C_1 \cup C_2) = 1 - P(C_1^c \cap C_2^c) = 1 - P(C_2^c) \cdot P(C_1^c) = 1 - (1 - P(C_2))(1 - P(C_2)) = 0.72$

$P(C_1 \cup C_2) = P(C_2) + P(C_1) - P(C_1 \cap C_2) = 0.72$ ■

Exercise 7 *Let* $f(x) = x/15$, $x = 1, 2, 3, 4, 5$, *zero elsewhere, be the p.d.f. of a* X*. Find* $\Pr(X = 1 \text{ or } 2)$, $\Pr\left(\frac{1}{2} < X < \frac{5}{2}\right)$, *and* $\Pr(1 \leq X \leq 2)$

Proof. $\Pr(1 \leq X \leq 2) = \Pr\left(\frac{1}{2} < X < \frac{5}{2}\right) = \Pr(X = 1 \text{ or } 2) = 1/15 + 2/15 = 1/5$ ■

Exercise 8 *Let the probability set function of the random variable X be*

$$P(A) = \int_A e^{-x} dx, \quad A \subset (0, \infty)$$

Let $A_k = \{x : 2 - 1/k < x \leq c\}$. $k = 1, 2, 3, \ldots$ Find $\lim_{k\to\infty} A_k$ and $P(\lim_{k\to\infty} A_k)$. Find $P(A_k)$ and $\lim_{k\to\infty} P(A_k)$

Proof. $A_\infty = \lim_{k\to\infty} A_k = \{x : 2 \leq x \leq c\}$
$P(A_\infty) = \int_{A_\infty} e^{-x} dx = \int_2^c e^{-x} dx = -e^{-x}|_2^c = e^{-2} - e^{-c}$
$P(A_k) = \int_{A_k} e^{-x} dx = \int_{2-1/k}^c e^{-x} dx = -e^{-x}|_{2-1/k}^c = e^{-2-1/k} - e^{-c}$
$\lim_{k\to\infty} P(A_k) = \lim_{k\to\infty} e^{-2-1/k} - e^{-c} = e^{-2} - e^{-c}$ ■

Exercise 9 *For each of the following probability density functions of X, compute* $\Pr(|X| < 1)$
(a) $f(x) = x^2/18$, $-3 < x < 3$, zero elsewhere
(b) $f(x) = (x+2)/18$, $-2 < x < 4$, zero elsewhere

Proof. $\Pr(|X| < 1) = \frac{1}{18} \int_{-1}^1 x^2 dx = \frac{x^3}{54}\Big|_{-1}^1 = \frac{1}{27}$
$\Pr(|X| < 1) = \frac{1}{18} \int_{-1}^1 (x+2) dx = (\frac{x^2}{36} + \frac{x}{9})\Big|_{-1}^1 = \frac{1-1}{36} + \frac{1+1}{9} = \frac{2}{9}$ ■

Exercise 10 *Let $f(x) = 1$, $0 < x < 1$, zero elsewhere, be the pdf of X. Find the distribution function and the pdf of $Y = \sqrt{X}$. Hint:* $\Pr(Y \leq y) = \Pr\left(\sqrt{X} \leq y\right) = \Pr(X \leq y^2)$

Proof. $F(x) = \int_{-\infty}^x f(t)dt = x$, *if* $0 < x < 1$, *1 if* $x > 1$, *0 if* $x < 0$
$F(y) = \Pr(X \leq y^2) = y^2$ *if* $0 < y < 1$, *1 if* $y > 1$, *0 if* $y < 0$
$f(y) = \frac{dF(y)}{dy} = 2y$, *if* $0 < y < 1$, *0 elsewhere* ■

Exercise 11 *Let $f(x) = 2x$, $0 < x < 1$, zero elsewhere, be the pdf of X. Find the distribution function and the pdf of $Y = X^2$.*

Proof. $F(x) = \int_{-\infty}^x f(t)dt = x^2$, *if* $0 < x < 1$, *1 if* $x > 1$, *0 if* $x < 0$
$F(y) = \Pr(X^2 \leq y) = \Pr(-\sqrt{y} \leq X \leq \sqrt{y}) = y$ *if* $0 < y < 1, y > 0$, *1 if* $y > 1$, *0 if* $y < 0$
$f(y) = \frac{dF(y)}{dy} = 1$, *if* $0 < y < 1$, *0 elsewhere* ■

Exercise 12 *Let X have the pdf $f(x) = (x+2)/18$, $-2 < x < 4$, zero elsewhere. Compute $E[X]$, and $E[X^2]$*

Proof. $E[X] = \frac{1}{18} \int_{-2}^4 x(x+2) dx = \frac{1}{18}(\frac{x^3}{3} + x^2)\Big|_{-2}^4 = \frac{1}{18}(\frac{4^3+2^3}{3} + 4^2 - 2^2) = 2$ ■

Proof. $E[X^2] = \frac{1}{18} \int_{-2}^4 x^2(x+2) dx = \frac{1}{18}(\frac{x^4}{4} + \frac{2x^3}{3})\Big|_{-2}^4 = \frac{1}{18}(\frac{4^4-2^4}{4} + \frac{128+16}{3}) = 6$ ■

Exercise 13 *Let $f(x) = \left(\frac{1}{2}\right)^x$, $x = 1, 2, 3, \ldots$, zero elsewhere, be the pdf of the random variable X. Find the moment generating function, the mean, and the variance of X*

Proof. $M(t) = E[\exp(t \cdot X)] = \sum \exp(\ nt)\left(\ \frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}e^t\right)}{1-\frac{1}{2}e^t} = \frac{1}{2e^{-t}-1}, -\ln 2 < t < \ln 2$
$E[X] = \frac{dM(t)}{dt}\Big|_{t=0} = \frac{2e^{-t}}{(2e^{-t}-1)^2}\Big|_{t=0} = \frac{2}{(2-1)^2} = 2$
$Var[X] = E[X^2] - E[X]^2 = \frac{d^2M(t)}{dt^2}\Big|_{t=0} - 4 = \frac{e^t(\frac{1}{2}+\frac{1}{4}e^t)}{(1-\frac{1}{2}e^t)^3}\Big|_{t=0} - 4 = \frac{3/4}{(1/2)^3} - 4 = 2$ ■

Exercise 14 *Let X be a random variable such that $E\left[(X-b)^2\right]$ exists for all real b. Show that $E\left[(X-b)^2\right]$ is a minimum when $b = E[X]$*

Proof. $E\left[(X-b)^2\right] \to \min_b$ *FOC:* $\frac{dE\left[(X-b)^2\right]}{db} = 2E[X-b] = 0$ *hence* $b^* = E[X]$ ■

Exercise 15 *Let X be a random variable with mean μ and variance σ^2. The ratio $E\left[(X-\mu)^3\right]/\sigma^3$ is often used as a measure of* skewness*. Compute the skewness for the following probability density functions:*
(a) $f(x) = (x+1)/2$, $-1 < x < 1$, *zero elsewhere*
(b) $f(x) = \frac{1}{2}$, $-1 < x < 1$, *zero elsewhere*
(c) $f(x) = (1-x)/2$, $-1 < x < 1$, *zero elsewhere*

Proof. *(a)* $\mu = \int_{-1}^{1} x\frac{(x+1)}{2}dx = 1/3$, $\sigma = \int_{-1}^{1}(x-\mu)^2\frac{(x+1)}{2}dx = \sqrt{2}/3$,
$E\left[(X-\mu)^3\right]\sigma^3 = \int_{-1}^{1}(x-\mu)^3\frac{(x+1)}{2\sigma^3}dx = -2\sqrt{2}/5$
(b) $\mu = \int_{-1}^{1} x\frac{1}{2}dx = 0$, $\sigma = \int_{-1}^{1}(x-\mu)^2\frac{1}{2}dx = 1/\sqrt{3}$,
$E\left[(X-\mu)^3\right]\sigma^3 = \int_{-1}^{1}(x-\mu)^3\frac{1}{2\sigma^3}dx = 0$
(c) $\mu = \int_{-1}^{1} x\frac{(1-x)}{2}dx = -1/3$, $\sigma = \int_{-1}^{1}(x-\mu)^2\frac{(1-x)}{2}dx = \sqrt{2}/3$,
$E\left[(X-\mu)^3\right]\sigma^3 = \int_{-1}^{1}(x-\mu)^3\frac{(1-x)}{2\sigma^3}dx = 2\sqrt{2}/5$ ■

Exercise 16 *Let X be a random variable with mean μ and variance σ^2. The ratio $E\left[(X-\mu)^4\right]/\sigma^4$ is often used as a measure of* kurtosis*. Compute the kurtosis for the following probability density functions:*
(a) $f(x) = \frac{1}{2}$, $-1 < x < 1$, *zero elsewhere*
(b) $f(x) = 3(1-x^2)/4$, $-1 < x < 1$, *zero elsewhere*

Proof. *(a)* $\mu = \int_{-1}^{1} x\frac{1}{2}dx = 0$, $\sigma = \int_{-1}^{1}(x-\mu)^2\frac{1}{2}dx = 1/\sqrt{3}$,
$E\left[(X-\mu)^4\right]\sigma^4 = \int_{-1}^{1}(x-\mu)^4\frac{1}{2\sigma^4}dx = 9/5$
(b) $\mu = \int_{-1}^{1} x\frac{3(1-x^2)}{4}dx = 0$, $\sigma = \int_{-1}^{1}(x-\mu)^2\frac{3(1-x^2)}{4}dx = 1/\sqrt{5}$,
$E\left[(X-\mu)^4\right]\sigma^4 = \int_{-1}^{1}(x-\mu)^4\frac{3(1-x^2)}{4\sigma^4}dx = 15/7$ ■

Exercise 17 *Let $\psi(t) = \ln M(t)$, where $M(t)$ is the moment generating function of a distribution. Prove that $\psi'(0) = \mu$ and $\psi''(0) = \sigma^2$*

Proof. $M(0) = E[\exp(t\cdot X)] = E[\exp(0)] = E[1] = 1$
$\psi'(0) = M'(0)/M(0) = E[X]/1 = \mu$
$\psi''(0) = M''(0)/M(0) - (M'(0)/M(0))^2 = E[X^2] - E[X]^2 = \sigma^2$