Exercise 1 If the sample space is \( \Omega = C_1 \cup C_2 \) and if \( P(C_1) = 0.8 \) and \( P(C_2) = 0.5 \), find \( P(C_1 \cap C_2) \).

**Proof.** \( P(C_1 \cap C_2) = P(C_1) + P(C_2) - P(C_1 \cup C_2) = 0.8 + 0.5 - 1 = 0.3 \)

Exercise 2 If \( P(C) > 0 \) and if \( C_1, C_2, \ldots \) are mutually disjoint sets, show that

\[
P(C_1 \cup C_2 \cup \cdots | C) = P(C_1 | C) + P(C_2 | C) + \cdots
\]

**Proof.** \( P(C_1 \cup C_2 \cup \cdots | C) = \frac{P(C_1 \cup C_2 \cup \cdots \cup C | P(C)}{P(C)} = \frac{P(C_1 \cup C_2 \cup \cdots | P(C)}{P(C)} = \frac{P(C_1 \cup C_2 \cup \cdots)}{P(C)} = P(C_1 | C) + P(C_2 | C) + \cdots \)

Exercise 3 Prove that

\[
P(C_1 \cap C_2 \cap C_3 \cap C_4) = P(C_1) P(C_2 | C_1) P(C_3 | C_1 \cap C_2) P(C_4 | C_1 \cap C_2 \cap C_3)
\]

**Proof.** \[ P(C_1 \cap C_2 \cap C_3 \cap C_4) = P(C_1) P(C_2 | C_1) P(C_3 | C_1 \cap C_2) P(C_4 | C_1 \cap C_2 \cap C_3) \]

Exercise 4 Bowl 1 contains 3 red chips and 7 blue chips. Bowl 2 contains 6 red chips and 4 blue chips. A bowl is selected at random and then 1 chip is drawn from this bowl.

(a) Compute the probability that this chip is red.

(b) Relative to the hypothesis that the chip is red, find the conditional probability that it is drawn from Bowl 2

**Proof.** Define \( A = \{1, 2\} - \) number of bowl, \( B = \{r, g\} - \) color of chip.

(a) \( P(B = r) = P(B = r \cap (A = 1 \cup A = 2)) = P(B = r \cap A = 1) + P(B = r \cap A = 2) = P(B = r | A = 1)P(A = 1) + P(B = r | A = 2)P(A = 2) = 3/10 \times 1/2 + 6/10 \times 1/2 = 9/20. \)

(b) \( P(A = 2 | B = r) = \frac{P(A = 2 \cap B = r)}{P(B = r)} = \frac{P(B = r | A = 2)P(A = 2)}{P(B = r)} = \frac{6/20}{9/20} = 2/3 \)

Exercise 5 If \( C_1 \) and \( C_2 \) are independent events, show that \( C_1 \) and \( C_2^c \) are also independent.

**Proof.** If \( C_1 \) and \( C_2 \) are independent events, then \( P(C_2 | C_1) = P(C_2) \), then \( P(C_2 | C_1) = P(C_1 \cap C_2^c) / P(C_1) = (P(C_1) - P(C_1 \cap C_2)) / P(C_1) = 1 - P(C_2 | C_1) = 1 - P(C_2) = P(C_2^c) \) which means that \( C_1 \) and \( C_2^c \) are independent events.

Exercise 6 Let \( C_1 \) and \( C_2 \) be independent events such that \( P(C_1) = 0.6 \) and \( P(C_2) = 0.3 \). Compute \( P(C_1 \cap C_2) \) and \( P(C_1 \cup C_2) \)

**Proof.** \[ P(C_1 \cap C_2) = P(C_2 | C_1) \cdot P(C_1) = P(C_2) \cdot P(C_1) = 0.18 \]

\[
P(C_1 \cup C_2) = 1 - P(C_1^c \cap C_2^c) = 1 - P(C_2^c) \cdot P(C_1^c) = 1 - (1 - P(C_2))(1 - P(C_1)) = 0.72
\]

\[
P(C_1 \cup C_2) = P(C_2) + P(C_1) - P(C_1 \cap C_2) = 0.72
\]

Exercise 7 Let \( f(x) = x/15, x = 1, 2, 3, 4, 5 \), zero elsewhere, be the p.d.f. of a \( X \). Find \( \Pr(X = 1 \text{ or } 2) \), \( \Pr(\frac{1}{2} < X < \frac{5}{2}) \), and \( \Pr(1 \leq X \leq 2) \).

**Proof.** \( \Pr(1 \leq X \leq 2) = \Pr(\frac{1}{2} < X < \frac{5}{2}) = \Pr(X = 1 \text{ or } 2) = 1/15 + 2/15 = 1/5 \)
Exercise 8 Let the probability set function of the random variable \( X \) be
\[
P(A) = \int_A e^{-x}dx, \quad A \subset (0, \infty)
\]

Let \( A_k = \{x : 2 - 1/k < x \leq c\}. \) \( k = 1, 2, 3, \ldots \) Find \( \lim_{k \to \infty} A_k \) and \( P(\lim_{k \to \infty} A_k) \). Find \( P(A_k) \) and \( \lim_{k \to \infty} P(A_k) \)

**Proof.** \( A_\infty = \lim_{k \to \infty} A_k = \{x : 2 \leq x \leq c\} \)
\[
P(A_\infty) = \int_{A_\infty} e^{-x}dx = \int_2^c e^{-x}dx = -e^{-x}\Big|_2^c = e^{-2} - e^{-c}
\]
\[
P(A_k) = \int_{A_k} e^{-x}dx = \int_{2-1/k}^{2} e^{-x}dx = -e^{-x}\Big|_{2-1/k}^{2} = e^{-2} - e^{-1/k} - e^{-c}
\]
\[
\lim_{k \to \infty} P(A_k) = \lim_{k \to \infty} e^{-2} - e^{-1/k} - e^{-c} = e^{-2} - e^{-c}
\]

Exercise 9 For each of the following probability density functions of \( X \), compute \( \Pr(|X| < 1) \)

(a) \( f(x) = x^2/18, -3 < x < 3, \) zero elsewhere

(b) \( f(x) = (x+2)/18, -2 < x < 4, \) zero elsewhere

**Proof.** \( \Pr(|X| < 1) = \frac{1}{18} \int_{-1}^{1} x^2dx = \frac{2}{27} \)
\[
\Pr(|X| < 1) = \frac{1}{18} \int_{-1}^{1} (x+2)dx = \left(\frac{x^2}{3} + \frac{x}{3}\right)\Big|_{-1}^{1} = \frac{1}{9} + \frac{2}{9} = \frac{2}{9}
\]

Exercise 10 Let \( f(x) = 1, 0 < x < 1, \) zero elsewhere, be the pdf of \( X \). Find the distribution function and the pdf of \( Y = \sqrt{X} \). Hint: \( \Pr(Y \leq y) = \Pr(\sqrt{X} \leq y) = \Pr(X \leq y^2) \)

**Proof.** \( F(x) = \int_{-\infty}^{x} f(t)dt = x, \) if \( 0 < x < 1, \) if \( x > 1, \) 0 if \( x < 0 \)
\[
F(y) = \Pr(X \leq y^2) = y^2 \text{ if } 0 < y < 1, \text{ if } y > 1, \text{ 0 if } y < 0
\]
\[
f(y) = \frac{dF(y)}{dy} = 2y, \text{ if } 0 < y < 1, \text{ 0 elsewhere}
\]

Exercise 11 Let \( f(x) = 2x, 0 < x < 1, \) zero elsewhere, be the pdf of \( X \). Find the distribution function and the pdf of \( Y = X^2 \).

**Proof.** \( F(x) = \int_{-\infty}^{x} f(t)dt = x^2, \) if \( 0 < x < 1, \) if \( x > 1, \) 0 if \( x < 0 \)
\[
F(y) = \Pr(X^2 \leq y) = \Pr(-\sqrt{y} \leq X \leq \sqrt{y}) = y \text{ if } 0 < y < 1, \text{ if } y > 0, \text{ 0 if } y < 0
\]
\[
f(y) = \frac{dF(y)}{dy} = 1, \text{ if } 0 < y < 1, \text{ 0 elsewhere}
\]

Exercise 12 Let \( X \) have the pdf \( f(x) = (x+2)/18, -2 < x < 4, \) zero elsewhere. Compute \( E[X] \), and \( E[X^2] \)

**Proof.** \( E[X] = \frac{1}{18} \int_{-2}^{2} x(x+2)dx = \frac{1}{18}\left(\frac{x^3}{3} + x^2\right)\Big|_{-2}^{2} = \frac{1}{18}\left(\frac{4^3+2^3}{3} + 4^2 - 2^2\right) = 2 \)

**Proof.** \( E[X^2] = \frac{1}{18} \int_{-2}^{2} x^2(x+2)dx = \frac{1}{18}\left(\frac{x^4}{4} + \frac{2x^3}{3}\right)\Big|_{-2}^{2} = \frac{1}{18}\left(\frac{4^4-2^4}{4} + \frac{128+16}{3}\right) = 6 \)

Exercise 13 Let \( f(x) = \left(\frac{1}{2}\right)^x, x = 1, 2, 3, \ldots, \) zero elsewhere, be the pdf of the random variable \( X \). Find the moment generating function, the mean, and the variance of \( X \)

**Proof.** \( M(t) = E[\exp(t \cdot X)] = \sum \exp(n(t)\left(\frac{1}{2}\right)^n) = \frac{(\frac{1}{2}e^t)}{1-\frac{1}{2}e^t} = \frac{1}{2e^t - 1}, - \ln 2 < t < \ln 2
\]
\[
E[X] = \frac{dM(t)}{dt}\Big|_{t=0} = \frac{2e^{-t}}{(2e^{t-1})^2} = \frac{2}{(2-1)^2} = 2
\]
\[
Var[X] = E[X^2] - E[X]^2 = \frac{d^2M(t)}{dt^2}\Big|_{t=0} - 4 = \frac{e^t(\frac{1}{2}+\frac{3}{4}e^t)}{(1-\frac{1}{2}e^t)^3}\Big|_{t=0} - 4 = \frac{3/4}{(1/2)^3} - 4 = 2
\]

Exercise 14 Let \( X \) be a random variable such that \( E[(X-b)^2] \) exists for all real \( b \). Show that \( E[(X-b)^2] \) is a minimum when \( b = E[X] \)

**Proof.** \( E[(X-b)^2] \rightarrow \min_b \text{ FOC: } \frac{dE[(X-b)^2]}{db} = 2E[X-b] = 0 \text{ hence } b^* = E[X] \)
Exercise 15 Let $X$ be a random variable with mean $\mu$ and variance $\sigma^2$. The ratio $E\left[(X - \mu)^3\right] / \sigma^3$ is often used as a measure of skewness. Compute the skewness for the following probability density functions:

(a) $f(x) = (x + 1)/2, -1 < x < 1$, zero elsewhere
(b) $f(x) = \frac{1}{2}, -1 < x < 1$, zero elsewhere
(c) $f(x) = (1 - x)/2, -1 < x < 1$, zero elsewhere

Proof. (a) $\mu = \int_{-1}^{1} x \frac{x+1}{2} dx = 1/3, \quad \sigma = \int_{-1}^{1} (x - \mu)^2 \frac{x+1}{2} dx = \sqrt{2}/3,$

$E[(X - \mu)^3] \sigma^3 = \int_{-1}^{1} (x - \mu)^3 \frac{x+1}{2} dx = -2\sqrt{2}/5$

(b) $\mu = \int_{-1}^{1} x \frac{1}{2} dx = 0, \quad \sigma = \int_{-1}^{1} (x - \mu)^2 \frac{1}{2} dx = 1/\sqrt{3},$

$E[(X - \mu)^3] \sigma^3 = \int_{-1}^{1} (x - \mu)^3 \frac{1}{2} dx = 0$

(c) $\mu = \int_{-1}^{1} x \frac{(1-x)}{2} dx = -1/3, \quad \sigma = \int_{-1}^{1} (x - \mu)^2 \frac{(1-x)}{2} dx = \sqrt{2}/3,$

$E[(X - \mu)^3] \sigma^3 = \int_{-1}^{1} (x - \mu)^3 \frac{(1-x)}{2} dx = 2\sqrt{2}/5$ ■

Exercise 16 Let $X$ be a random variable with mean $\mu$ and variance $\sigma^2$. The ratio $E\left[(X - \mu)^4\right] / \sigma^4$ is often used as a measure of kurtosis. Compute the kurtosis for the following probability density functions:

(a) $f(x) = \frac{1}{2}, -1 < x < 1$, zero elsewhere
(b) $f(x) = \frac{3}{4}(1 - x^2)/4, -1 < x < 1$, zero elsewhere

Proof. (a) $\mu = \int_{-1}^{1} x \frac{1}{2} dx = 0, \quad \sigma = \int_{-1}^{1} (x - \mu)^2 \frac{1}{2} dx = 1/\sqrt{3},$

$E[(X - \mu)^4] \sigma^4 = \int_{-1}^{1} (x - \mu)^4 \frac{1}{2} dx = 9/5$

(b) $\mu = \int_{-1}^{1} x \frac{3(1-x^2)}{4} dx = 0, \quad \sigma = \int_{-1}^{1} (x - \mu)^2 \frac{3(1-x^2)}{4} dx = 1/\sqrt{5},$

$E[(X - \mu)^4] \sigma^4 = \int_{-1}^{1} (x - \mu)^4 \frac{3(1-x^2)}{4} dx = 15/7$ ■

Exercise 17 Let $\psi(t) = \ln M(t)$, where $M(t)$ is the moment generating function of a distribution. Prove that $\psi'(0) = \mu$ and $\psi''(0) = \sigma^2$

Proof. $M(0) = E[\exp(t \cdot X)] = E[\exp(0)] = E[1] = 1$

$\psi'(0) = M'(0) / M(0) = E[X] / 1 = \mu$

$\psi''(0) = M''(0) / M(0) - (M'(0) / M(0))^2 = E[X^2] - E[X]^2 = \sigma^2$