Econ202C Final Exam Anton Cheremukhin

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Exercise 1 *Money and Capital*

(a) Social planner's problem: $\Sigma_{t=0}^{\infty}\beta^t\left(u\left(c_t^e\right)+u\left(c_t^o\right)\right)\rightarrow\max_{c_t^i,k_{t+1}^i}$
s.t. $c_t^e+c_t^o+k_{t+1}^e+k_{t+1}^o\leq A_t^e f\left(k_t^e\right)+A_t^o f\left(k_t^o\right)$ s.t. $A_0^o f\left(k_0^o\right)=y^o, A_0^e f\left(k_0^e\right)=y^e$
$L=\Sigma_{t=0}^{\infty}\beta^t\left\{u\left(c_t^e\right)+u\left(c_t^o\right)+\lambda_t\left[A_t^e f\left(k_t^e\right)+A_t^o f\left(k_t^o\right)-c_t^e-c_t^o-k_{t+1}^e-k_{t+1}^o\right]\right\}$
$\text{FOC}_{c_t^e}$: $u'\left(c_t^e\right)-\lambda_t=0$ $\text{FOC}_{k_{t+1}^e}$: $-\beta^t\lambda_t+\beta^{t+1}\lambda_{t+1}A_{t+1}^e f'\left(k_{t+1}^e\right)=0$
$\text{FOC}_{c_t^o}$: $u'\left(c_t^o\right)-\lambda_t=0$ $\text{FOC}_{k_{t+1}^o}$: $-\beta^t\lambda_t+\beta^{t+1}\lambda_{t+1}A_{t+1}^o f'\left(k_{t+1}^o\right)=0$
Therefore, $\frac{\lambda_t}{\lambda_{t+1}}=\beta A_{t+1}^o f'\left(k_{t+1}^o\right)=\beta A_{t+1}^e f'\left(k_{t+1}^e\right)=\frac{u'\left(c_t^e\right)}{u'\left(c_{t+1}^e\right)}=\frac{u'\left(c_t^o\right)}{u'\left(c_{t+1}^o\right)}$, $u'\left(c_t^e\right)=u'\left(c_t^o\right)$.
We also have a resource constraint: $c_t^e+c_t^o+k_{t+1}^e+k_{t+1}^o=A_t^e f\left(k_t^e\right)+A_t^o f\left(k_t^o\right)$.
We have five equations in five unknowns, and the initial conditions. This completely defines the paths of all variables. Steady-state:
Imagine we are in an even period, the even guy has high yield and the odd guy has low yield.
$\beta\underline{A}f'\left(k_{t+1}^o\right)=\beta\overline{A}f'\left(k_{t+1}^e\right)=\frac{u'\left(c_t^e\right)}{u'\left(c_{t+1}^e\right)}=\frac{u'\left(c_t^o\right)}{u'\left(c_{t+1}^o\right)}$ $c_t^e+c_t^o+k_{t+1}^e+k_{t+1}^o=\overline{A}f\left(k_t^e\right)+\underline{A}f\left(k_t^o\right)$
A similar constraint we shall have in the next period. Inada conditions on the utility assure that we always want to increase consumption. Inada conditions on the production function assure that we would not like to increase consumption indefinitely, but there will be a steady state. The discounting property of the utility implies that in a steady state consumption of both guys will be constant, and hence the rates of return on both yields will be equalized: $\underline{A}f'\left(k_{t+1}^o\right)=\overline{A}f'\left(k_{t+1}^e\right)=\frac{1}{\beta}$ - it is optimal to invest more in more efficient capital. So, in the steady-state optimal capital will be moving between two values: high and low, consumption of both will be constant (the same because $u'\left(c_t^e\right)=u'\left(c_t^o\right)$).
Optimal s-s allocations: $\left\{\overline{k},\underline{k},c\right\}:\underline{A}f'\left(\overline{k}\right)=\overline{A}f'\left(\overline{k}\right)=\frac{1}{\beta}$, $2c+\overline{k}+\underline{k}=\overline{A}f\left(\overline{k}\right)+\underline{A}f\left(\underline{k}\right)$
(b) Individual problem with bonds (1): $\Sigma_{t=0}^{\infty}\beta^t\left(u\left(c_t^i\right)\right)\rightarrow\max_{c_t^i,k_{t+1}^i,b_{t+1}^i}$
s.t. $c_t^i+k_{t+1}^i+q_t b_{t+1}^i\leq A_t^i f\left(k_t^i\right)+b_t^i$ s.t. $A_0^i f\left(k_0^i\right)=y^i$, b_0^i
Market clearing (2): $c_t^e+c_t^o+k_{t+1}^e+k_{t+1}^o=A_t^e f\left(k_t^e\right)+A_t^o f\left(k_t^o\right)$, $b_{t+1}^e+b_{t+1}^o=0$
Equilibrium: allocations $\left\{c_t^i,k_{t+1}^i,b_{t+1}^i\right\}$ and prices $\{1,q_t\}$ s.t. (1) given p and (2).
$L=\Sigma_{t=0}^{\infty}\beta^t\left\{u\left(c_t^i\right)+\lambda_t^i\left[A_t^i f\left(k_t^i\right)+b_t^i-c_t^i-k_{t+1}^i-q_t b_{t+1}^i\right]\right\}$
$\text{FOC}_{c_t^i}$: $u'\left(c_t^e\right)-\lambda_t^i=0$ $\text{FOC}_{k_{t+1}^i}$: $-\beta^t\lambda_t^i+\beta^{t+1}\lambda_{t+1}^i A_{t+1}^i f'\left(k_{t+1}^i\right)=0$
$\text{FOC}_{b_{t+1}^i}$: $-\beta^t q_t\lambda_t^i+\beta^{t+1}\lambda_{t+1}^i=0$
Therefore, $\frac{\lambda_t^i}{\lambda_{t+1}^i}=\frac{\beta}{q_t}=\beta A_{t+1}^i f'\left(k_{t+1}^i\right)=\frac{u'\left(c_t^i\right)}{u'\left(c_{t+1}^i\right)}$, $u'\left(c_t^i\right)=\lambda_t^i$.
Through the bond market the rates of return and the lagrange multipliers are equalized:
$\beta A_{t+1}^e f'\left(k_{t+1}^e\right)=\frac{\beta}{q_t}=\beta A_{t+1}^o f'\left(k_{t+1}^o\right)$ $\lambda_t^e=\lambda_t^o=\lambda_t$
From $u'\left(c_t^e\right)=\lambda_t=u'\left(c_t^o\right)$ we again have the same constant consumption in steady-state. This pins down q_t. The resource constraint (market clearing) has not changed: $c_t^e+c_t^o+k_{t+1}^e+k_{t+1}^o=A_t^e f\left(k_t^e\right)+A_t^o f\left(k_t^o\right)$.
The trading arrangement with bonds gives the same equations, and therefore the same allocations, as in the central planner's problem. The steady-state price is: $q=\beta$. In the steady-state in an even period the budget constraints are: $\overline{A}f\left(\overline{k}\right)+b_t^e=c+\underline{k}+\beta b_{t+1}^e$ $\underline{A}f\left(\underline{k}\right)+b_t^o=c+\overline{k}+\beta b_{t+1}^o$.

From symmetry $b_{t+1}^o = b_t^e = \underline{b}$, $b_{t+1}^e = b_t^o = \overline{b}$. The bond allocations will also be moving between two values:

$$\begin{cases} \overline{A}f\left(\overline{k}\right) + \underline{b} = c + \underline{k} + \beta\overline{b} \\ \underline{A}f\left(\underline{k}\right) + \overline{b} = c + \overline{k} + \beta\underline{b} \end{cases} \Leftrightarrow \begin{cases} \underline{b} = \left[c\left(1+\beta\right) + \beta\overline{k} + \underline{k} - \overline{A}f\left(\overline{k}\right) - \beta\underline{A}f\left(\underline{k}\right)\right] / \left(1-\beta^2\right) \\ \overline{b} = \left[c\left(1+\beta\right) + \beta\underline{k} + \overline{k} - \underline{A}f\left(\underline{k}\right) - \beta\overline{A}f\left(\overline{k}\right)\right] / \left(1-\beta^2\right) \end{cases}$$

type of period	even	odd	even	odd	...
odd endowment	$\underline{A}f\left(\underline{k}\right)$	$\overline{A}f\left(\overline{k}\right)$	$\underline{A}f\left(\underline{k}\right)$	$\overline{A}f\left(\overline{k}\right)$	...
odd allocation	$c, \overline{k}, \underline{b}$	$c, \underline{k}, \overline{b}$	$c, \overline{k}, \underline{b}$	$c, \underline{k}, \overline{b}$	...
even endowment	$\overline{A}f\left(\overline{k}\right)$	$\underline{A}f\left(\underline{k}\right)$	$\overline{A}f\left(\overline{k}\right)$	$\underline{A}f\left(\underline{k}\right)$	...
even allocation	$c, \underline{k}, \overline{b}$	$c, \overline{k}, \underline{b}$	$c, \underline{k}, \overline{b}$	$c, \overline{k}, \underline{b}$	...

(c) I assume agents are of measure 1. The same economy with money but without bonds:

Individual problem with money (1): $\Sigma_{t=0}^{\infty}\beta^t\left(u\left(c_t^i\right)\right) \rightarrow \max_{c_t^i, k_{t+1}^i, m_{t+1}^i}$

s.t. $c_t^i + k_{t+1}^i + \frac{m_{t+1}^i}{p_{t+1}}\frac{p_{t+1}}{p_t} \leq A_t^i f\left(k_t^i\right) + \frac{m_t^i}{p_t}$, $m_{t+1}^i \geq 0$, s.t. $A_0^i f\left(k_0^i\right) = y^i$, b_0^i

Market clearing (2): $c_t^e + c_t^o + k_{t+1}^e + k_{t+1}^o = A_t^e f\left(k_t^e\right) + A_t^o f\left(k_t^o\right)$, $m_{t+1}^e + m_{t+1}^o = M$

Equilibrium: allocations $\left\{c_t^i, k_{t+1}^i, m_{t+1}^i\right\}$ and prices $\{1, p_t\}$ s.t. (1) given p and (2).

$L = \Sigma_{t=0}^{\infty}\beta^t\left\{u\left(c_t^i\right) + \lambda_t^i\left[A_t^i f\left(k_t^i\right) + \frac{m_t^i}{p_t} - c_t^i - k_{t+1}^i - \frac{m_{t+1}^i}{p_{t+1}}\frac{p_{t+1}}{p_t}\right] + \mu_t^i m_{t+1}^i\right\}$

FOC$_{c_t^i}$: $u'\left(c_t^e\right) - \lambda_t^i = 0$ FOC$_{k_{t+1}^i}$: $-\beta^t\lambda_t^i + \beta^{t+1}\lambda_{t+1}^i A_{t+1}^i f'\left(k_{t+1}^i\right) = 0$

FOC$_{m_{t+1}^i}$: $-\beta^t\frac{1}{p_t}\lambda_t^i + \beta^t\mu_t^i + \beta^{t+1}\frac{1}{p_{t+1}}\lambda_{t+1}^i = 0$

Therefore, $\frac{\lambda_t^i}{\lambda_{t+1}^i} = \beta A_{t+1}^i f'\left(k_{t+1}^i\right) = \frac{u'\left(c_t^i\right)}{u'\left(c_{t+1}^i\right)}$, $u'\left(c_t^i\right) = \lambda_t^i$, $\frac{1}{p_t}\lambda_t^i = \mu_t^i + \beta\frac{1}{p_{t+1}}\lambda_{t+1}^i$

The last eqation simplifies to $\frac{\lambda_t^i}{\lambda_{t+1}^i} = \beta\frac{p_t}{p_{t+1}}$ if the constraint is not binding and $m_{t+1}^i = 0$ if it is binding. In our setting the poor guy will have the constraint binding. In an even period:

$\frac{\lambda_t^e}{\lambda_{t+1}^e} = \beta\underline{A}_{t+1}^e f'\left(k_{t+1}^e\right) = \frac{u'\left(c_t^e\right)}{u'\left(c_{t+1}^e\right)} = \beta\frac{p_t}{p_{t+1}}$ $m_{t+1}^e = M$ $\overline{A}_t^e f\left(k_t^e\right) = c_t^e + k_{t+1}^e + \frac{M}{p_t}$

$\frac{\lambda_t^o}{\lambda_{t+1}^o} = \beta\overline{A}_{t+1}^o f'\left(k_{t+1}^o\right) = \frac{u'\left(c_t^o\right)}{u'\left(c_{t+1}^o\right)}$ $m_{t+1}^o = 0$ $\underline{A}_t^o f\left(k_t^o\right) + \frac{M}{p_t} = c_t^o + k_{t+1}^o$ $\frac{\lambda_t^o}{\lambda_{t+1}^o} > \beta\frac{p_t}{p_{t+1}}$

These are nine equations in nine unknowns, which together with the initial conditions describe the dynamics. **(d)** Symmetric steady-state: $c_t^e = c_{t+1}^o = \overline{c}$, $c_t^o = c_{t+1}^e = \underline{c}$, $k_{t+1}^e = k_t^o = \underline{k'}$, $k_{t+1}^o = k_t^e = \overline{k'}$, $m_{t+1}^e = m_t^o = M$, $m_{t+1}^o = m_t^e = 0$, $p_t = p_{t+1} = p$.

Equations simplify to: $\beta\underline{A}f'\left(\underline{k'}\right) = \frac{u'(\overline{c})}{u'(\underline{c})} = \beta\frac{p_t}{p_{t+1}} = \beta$ $\overline{A}f\left(\overline{k'}\right) = \overline{c} + \underline{k'} + \frac{M}{p}$

$\beta\overline{A}f'\left(\overline{k'}\right) = \frac{u'(\underline{c})}{u'(\overline{c})}$ $\underline{A}f\left(\underline{k'}\right) + \frac{M}{p} = \underline{c} + \overline{k'}$ Verify that $\frac{\lambda_t^o}{\lambda_{t+1}^o} = \frac{1}{\beta} > \beta\frac{p_t}{p_{t+1}} = \beta$

Therefore, $\beta\underline{A}f'\left(\underline{k'}\right) = \beta$ $\beta\overline{A}f'\left(\overline{k'}\right) = \frac{1}{\beta}$ $\frac{M}{p} = \overline{A}f\left(\overline{k'}\right) - \overline{c} - \underline{k'}$

$\underline{A}f\left(\underline{k'}\right) + \overline{A}f\left(\overline{k'}\right) = \underline{c} + \overline{k'} + \overline{c} + \underline{k'}$ $\frac{u'(\overline{c})}{u'(\underline{c})} = \beta$ - five equations in five unknowns.

type of period	even	odd	even	odd	...
odd endowment	$\underline{A}f\left(\underline{k'}\right), M$	$\overline{A}f\left(\overline{k'}\right), 0$	$\underline{A}f\left(\underline{k'}\right), M$	$\overline{A}f\left(\overline{k'}\right), 0$	...
odd allocation	$\underline{c}, \overline{k'}, 0$	$\overline{c}, \underline{k'}, M$	$\underline{c}, \overline{k'}, 0$	$\overline{c}, \underline{k'}, M$	...
even endowment	$\overline{A}f\left(\overline{k'}\right), 0$	$\underline{A}f\left(\underline{k'}\right), M$	$\overline{A}f\left(\overline{k'}\right), 0$	$\underline{A}f\left(\underline{k'}\right), M$	...
even allocation	$\overline{c}, \underline{k'}, M$	$\underline{c}, \overline{k'}, 0$	$\overline{c}, \underline{k'}, M$	$\underline{c}, \overline{k'}, 0$	...

(e) Through the money market the rates of return and the lagrange multipliers are not equalized. Because of that we have both capitals and consumptions change over time. Hence, the monetary equilibrium does not in general implement the social planner's allocation, which is first best. Because of the non-negativity constaint on money holdings we get a second best solution.

Exercise 2 *Cash-in-Advance Model*

VERSION 1 of CIA constraint: the one that follows from the question.

a) (1) Household: $\Sigma_{t=0}^{\infty}\beta^t\left(u\left(c_{1t},c_{2t}\right)-v\left(L_t\right)\right)\rightarrow\max_{c_{1t},c_{2t},L_t,k_{1t+1},k_{2t+1},M_t}$

s.t. $p_{1t}c_{1t}+p_{2t}c_{2t}+p_{2t}\left(k_{1t+1}-\left(1-\delta\right)k_{1t}+k_{2t+1}-\left(1-\delta\right)k_{2t}\right)+M_t\leq$

$\leq r_{1t}k_{1t}+r_{2t}k_{2t}+w_tL_t+\left(1+\mu\right)M_{t-1},$ s.t. $\boxed{p_{1t}c_{1t}\leq\left(1+\mu\right)M_{t-1}}$ s.t. k_{10},k_{20},M_{-1}

(2) Firm i: $p_{it}y_{it}-r_{it}\widetilde{k}_{it}-w_tl_{it}\rightarrow\max_{k_{it},l_{it}}$ s.t. $y_{it}=f\left(\widetilde{k}_{it},l_{it}\right)$ $i=1,2$ $t=0,1...$

(3) Market clearing: $c_{1t}=y_{1t},$ $c_{2t}+\left(k_{1t+1}-\left(1-\delta\right)k_{1t}+k_{2t+1}-\left(1-\delta\right)k_{2t}\right)=y_{2t}$

$\widetilde{k}_{1t}=k_{1t},$ $\widetilde{k}_{2t}=k_{2t},$ $L_t=l_{1t}+l_{2t},$ $M_t=\left(1+\mu\right)M_{t-1}$

Competitive Equilibrium:

(i) allocations $\left\{c_{1t},c_{2t},k_{1t+1},k_{2t+1},M_t,L_t,\widetilde{k}_{1t},\widetilde{k}_{2t},l_{1t},l_{2t}\right\}_{t=0}^{\infty}$

(ii) prices $\left\{p_{1t},p_{2t},r_{1t},r_{2t},w_t,1\right\}_{t=0}^{\infty}$ such that

a) (i) solves (1) and (2) given (ii) b) (i) and (ii) satisfy (iii)

$$\textbf{b)}\ L=\Sigma_{t=0}^{\infty}\beta^t\left\{\begin{array}{c}u\left(c_{1t},c_{2t}\right)-v\left(L_t\right)+\psi_t\left[\left(1+\mu\right)M_{t-1}-p_{1t}c_{1t}\right]+\\+\lambda_t\left[\begin{array}{c}r_{1t}k_{1t}+r_{2t}k_{2t}+w_tL_t+\left(1+\mu\right)M_{t-1}-p_{1t}c_{1t}-p_{2t}c_{2t}-\\-p_{2t}\left(k_{1t+1}-\left(1-\delta\right)k_{1t}+k_{2t+1}-\left(1-\delta\right)k_{2t}\right)-M_t\end{array}\right]\end{array}\right\}$$

$\text{FOC}_{c_{1t}}$: $u_1'\left(c_{1t},c_{2t}\right)-\psi_tp_{1t}-\lambda_tp_{1t}=0$

$\text{FOC}_{c_{2t}}$: $u_2'\left(c_{1t},c_{2t}\right)-\lambda_tp_{2t}=0$

FOC_{L_t}: $-v'\left(L_t\right)+\lambda_tw_t=0$

$\text{FOC}_{k_{1t+1}}$: $\beta\lambda_{t+1}\left(r_{1t+1}+p_{2t+1}\left(1-\delta\right)\right)-\lambda_tp_{2t}=0$

$\text{FOC}_{k_{2t+1}}$: $\beta\lambda_{t+1}\left(r_{2t+1}+p_{2t+1}\left(1-\delta\right)\right)-\lambda_tp_{2t}=0$

FOC_{M_t}: $\psi_{t+1}\beta\left(1+\mu\right)+\lambda_{t+1}\beta\left(1+\mu\right)-\lambda_t=0$

$\left(1+\mu\right)M_{t-1}-p_{1t}c_{1t}=0$ $r_{1t}k_{1t}+r_{2t}k_{2t}+w_tL_t+\left(1+\mu\right)M_{t-1}=$

$=p_{1t}c_{1t}+p_{2t}c_{2t}+p_{2t}\left(k_{1t+1}-\left(1-\delta\right)k_{1t}+k_{2t+1}-\left(1-\delta\right)k_{2t}\right)+M_t$

$\text{FOC}_{l_{1t}}$: $p_{1t}f_l'\left(\widetilde{k}_{1t},l_{1t}\right)-w_t=0$

$\text{FOC}_{\widetilde{k}_{1t}}$: $p_{1t}f_k'\left(\widetilde{k}_{1t},l_{1t}\right)-r_{1t}=0$

$\text{FOC}_{l_{2t}}$: $p_{2t}f_l'\left(\widetilde{k}_{2t},l_{2t}\right)-w_t=0$

$\text{FOC}_{\widetilde{k}_{2t}}$: $p_{2t}f_k'\left(\widetilde{k}_{2t},l_{2t}\right)-r_{2t}=0$

$c_{1t}=f\left(\widetilde{k}_{1t},l_{1t}\right),$ $c_{2t}+\left(k_{1t+1}-\left(1-\delta\right)k_{1t}+k_{2t+1}-\left(1-\delta\right)k_{2t}\right)=f\left(\widetilde{k}_{2t},l_{2t}\right)$

$\widetilde{k}_{1t}=k_{1t},$ $\widetilde{k}_{2t}=k_{2t},$ $L_t=l_{1t}+l_{2t},$ $M_t=\left(1+\mu\right)M_{t-1}$

We have 18 equations in 17 unknowns given the initial conditions, one is for Walras Law.

c) Simplify: $r_{1t}=r_{2t}\equiv r_t$ $u_2'\left(c_{1t},c_{2t}\right)=p_{2t}\lambda_t$ $v'\left(L_t\right)=w_t\lambda_t$

$\beta\lambda_{t+1}p_{2t+1}\left(\frac{r_{t+1}}{p_{2t+1}}+\left(1-\delta\right)\right)=\lambda_tp_{2t}$ $\left(1+\mu\right)M_{t-1}=p_{1t}c_{1t}=M_t$

$p_{1t}f_l'\left(k_{1t},l_{1t}\right)=w_t=p_{2t}f_l'\left(k_{2t},l_{2t}\right)$ $p_{1t}f_k'\left(k_{1t},l_{1t}\right)=r_t=p_{2t}f_k'\left(k_{2t},l_{2t}\right)$

$r_t\left(k_{1t}+k_{2t}\right)+w_tL_t=p_{1t}c_{1t}+p_{2t}c_{2t}+p_{2t}\left(k_{1t+1}-\left(1-\delta\right)k_{1t}+k_{2t+1}-\left(1-\delta\right)k_{2t}\right)$

$c_{1t}=f\left(k_{1t},l_{1t}\right),$ $c_{2t}+\left(k_{1t+1}-\left(1-\delta\right)k_{1t}+k_{2t+1}-\left(1-\delta\right)k_{2t}\right)=f\left(k_{2t},l_{2t}\right)$ $L_t=l_{1t}+l_{2t}$

$u_1'\left(c_{1t},c_{2t}\right)-\psi_tp_{1t}-\lambda_tp_{1t}=0$ $\psi_{t+1}\beta\left(1+\mu\right)+\lambda_{t+1}\beta\left(1+\mu\right)-\lambda_t=0$

Steady-state: $k_1,k_2,c_1,c_2,l_1,l_2,L=const$ $p_{2t+1}/p_{2t}=1+\pi$ $M_t/p_{2t}=m$

Define : $\lambda_tp_{2t}=\lambda,$ $w_t/p_{2t}=w,$ $p_{1t}/p_{2t}=p,$ $r_t/p_{2t}=r,$ $\psi_tp_{2t}=\psi$

Therefore, $\left(1+\mu\right)\frac{M_{t-1}}{p_{2t-1}}\frac{p_{2t-1}}{p_{1t-1}}\frac{p_{1t-1}}{p_{1t}}=c_{1t}=\frac{M_t}{p_{2t}}\frac{p_{2t}}{p_{1t}}$ $\Leftrightarrow$ $\left(1+\mu\right)m\frac{1}{1+\pi}=pc_1=m$

$u_1'\left(c_1,c_2\right)-\psi p-\lambda p=0$ $\frac{p_{2t+1}\psi_{t+1}}{p_{2t+1}}\beta\left(1+\mu\right)+\frac{\lambda_{t+1}p_{2t+1}}{p_{2t+1}}\beta\left(1+\mu\right)-\frac{\lambda_tp_{2t}}{p_{2t}}=0\Leftrightarrow\psi\beta+\lambda\beta=\lambda$

Therefore, $\psi = \lambda\left(\frac{1}{\beta} - 1\right) \qquad u_1'(c_1, c_2) = \lambda p/\beta$.

Then the s-s is described by:

$u_2'(c_1, c_2) = \lambda \qquad u_1'(c_1, c_2) = \lambda p/\beta \qquad v'(L) = w\lambda \qquad \beta(r + (1-\delta)) = 1 \qquad \psi = \lambda\left(\frac{1}{\beta} - 1\right)$

$pf_l'(k_1, l_1) = w = f_l'(k_2, l_2) \qquad pf_k'(k_1, l_1) = r = f_k'(k_2, l_2) \qquad L = l_1 + l_2$

$r(k_1 + k_2) + wL = pc_1 + c_2 + \delta k_1 + \delta k_2 \qquad c_1 = f(k_1, l_1), \qquad c_2 + \delta k_1 + \delta k_2 = f(k_2, l_2)$

$\pi = \mu \qquad pc_1 = m$ — 15 equations in 14 unknowns.

Solution:

$\lambda = u_2' \qquad u_1' = \lambda p/\beta \qquad w = \frac{v'}{u'} \qquad r = \frac{1}{\beta} - (1-\delta) \qquad p = \frac{f_k'(k_2,l_2)}{f_k'(k_1,l_1)}$

$\pi = \mu \qquad m = \frac{r}{f_k'(k_{1t},l_{1t})}c_1 \qquad pf_l'(k_1, l_1) = \frac{v'(l_1+l_2)}{u_2'(c_1,c_2)} = f_l'(k_2, l_2) \qquad r = f_k'(k_2, l_2)$

$\frac{f_k'(k_2,l_2)}{f_k'(k_1,l_1)}u_2'(c_1, c_2) = \beta u_1'(c_1, c_2) \qquad c_1 = f(k_1, l_1) \qquad c_2 + \delta(k_1 + k_2) = f(k_2, l_2)$

Implied from the rest and CRS: $(r - \delta)(k_1 + k_2) + \frac{v'(l_1+l_2)}{u_2'(c_1,c_2)}(l_1 + l_2) = \frac{r}{f_k'(k_1,l_1)}c_1 + c_2$.

Now we have 6 equations in 6 unknowns - all real variables:

$\frac{f_k'(k_2,l_2)}{f_k'(k_1,l_1)}f_l'(k_1, l_1) = \frac{v'(l_1+l_2)}{u_2'(c_1,c_2)} = f_l'(k_2, l_2) \qquad \left(\frac{1}{\beta} - (1-\delta)\right) = f_k'(k_2, l_2)$

$\frac{f_k'(k_2,l_2)}{f_k'(k_1,l_1)}u_2'(c_1, c_2) = \beta u_1'(c_1, c_2) \qquad c_1 = f(k_1, l_1) \qquad c_2 + \delta(k_1 + k_2) = f(k_2, l_2)$

(d) This implies, that μ has no effect on real variables: consumption, capital and labor.

It also follows that money growth is directly reflected in the price level: $\pi = \mu$. All the price ratios remain the same. Money is superneutral here.

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VERSION 2 of CIA constraint: the one that probably was expected.

a) (1) Household: $\Sigma_{t=0}^{\infty}\beta^t(u(c_{1t}, c_{2t}) - v(L_t)) \to \max_{c_{1t},c_{2t},L_t,k_{1t+1},k_{2t+1},M_t}$

s.t. $p_{1t}c_{1t} + p_{2t}c_{2t} + p_{2t}(k_{1t+1} - (1-\delta)k_{1t} + k_{2t+1} - (1-\delta)k_{2t}) + M_t \leq$

$\leq r_{1t}k_{1t} + r_{2t}k_{2t} + w_tL_t + (1+\mu)M_{t-1}, \qquad$ s.t. $\boxed{p_{1t}c_{1t} \leq M_{t-1}} \qquad$ s.t. k_{10}, k_{20}, M_{-1}

(2) Firm i: $p_{it}y_{it} - r_{it}\widetilde{k}_{it} - w_tl_{it} \to \max_{k_{it},l_{it}} \qquad$ s.t. $y_{it} = f\left(\widetilde{k}_{it}, l_{it}\right) \qquad i = 1, 2 \qquad t = 0, 1...$

(3) Market clearing: $\qquad c_{1t} = y_{1t}, \qquad c_{2t} + (k_{1t+1} - (1-\delta)k_{1t} + k_{2t+1} - (1-\delta)k_{2t}) = y_{2t}$

$\widetilde{k}_{1t} = k_{1t}, \qquad \widetilde{k}_{2t} = k_{2t}, \qquad L_t = l_{1t} + l_{2t}, \qquad M_t = (1+\mu)M_{t-1}$

Competitive Equilibrium:

(i) allocations $\left\{c_{1t}, c_{2t}, k_{1t+1}, k_{2t+1}, M_t, L_t, \widetilde{k}_{1t}, \widetilde{k}_{2t}, l_{1t}, l_{2t}\right\}_{t=0}^{\infty}$

(ii) prices $\{p_{1t}, p_{2t}, r_{1t}, r_{2t}, w_t, 1\}_{t=0}^{\infty}$ such that

a) (i) solves (1) and (2) given (ii) b) (i) and (ii) satisfy (iii)

$$\textbf{b)}\ L = \Sigma_{t=0}^{\infty}\beta^t\left\{\begin{array}{c} u(c_{1t}, c_{2t}) - v(L_t) + \psi_t[M_{t-1} - p_{1t}c_{1t}] + \\ +\lambda_t\left[\begin{array}{c} r_{1t}k_{1t} + r_{2t}k_{2t} + w_tL_t + (1+\mu)M_{t-1} - p_{1t}c_{1t} - p_{2t}c_{2t} - \\ -p_{2t}(k_{1t+1} - (1-\delta)k_{1t} + k_{2t+1} - (1-\delta)k_{2t}) - M_t \end{array}\right]\end{array}\right\}$$

FOC$_{c_{1t}}$: $\quad u_1'(c_{1t}, c_{2t}) - \psi_tp_{1t} - \lambda_tp_{1t} = 0$

FOC$_{c_{2t}}$: $\quad u_2'(c_{1t}, c_{2t}) - \lambda_tp_{2t} = 0$

FOC$_{L_t}$: $\quad -v'(L_t) + \lambda_tw_t = 0$

FOC$_{k_{1t+1}}$: $\quad \beta\lambda_{t+1}(r_{1t+1} + p_{2t+1}(1-\delta)) - \lambda_tp_{2t} = 0$

FOC$_{k_{2t+1}}$: $\quad \beta\lambda_{t+1}(r_{2t+1} + p_{2t+1}(1-\delta)) - \lambda_tp_{2t} = 0$

FOC$_{M_t}$: $\quad \psi_{t+1}\beta + \lambda_{t+1}\beta(1+\mu) - \lambda_t = 0$

$(1+\mu)M_{t-1} - p_{1t}c_{1t} = 0 \qquad r_{1t}k_{1t} + r_{2t}k_{2t} + w_tL_t + (1+\mu)M_{t-1} =$

$= p_{1t}c_{1t} + p_{2t}c_{2t} + p_{2t}(k_{1t+1} - (1-\delta)k_{1t} + k_{2t+1} - (1-\delta)k_{2t}) + M_t$

FOC$_{l_{1t}}$: $\quad p_{1t}f_l'\left(\widetilde{k}_{1t}, l_{1t}\right) - w_t = 0$

FOC$_{\widetilde{k}_{1t}}$: $\quad p_{1t} f'_k\left(\widetilde{k}_{1t}, l_{1t}\right) - r_{1t} = 0$

FOC$_{l_{2t}}$: $\quad p_{2t} f'_l\left(\widetilde{k}_{2t}, l_{2t}\right) - w_t = 0$

FOC$_{\widetilde{k}_{2t}}$: $\quad p_{2t} f'_k\left(\widetilde{k}_{2t}, l_{2t}\right) - r_{2t} = 0$

$c_{1t} = f\left(\widetilde{k}_{1t}, l_{1t}\right), \qquad c_{2t} + (k_{1t+1} - (1-\delta) k_{1t} + k_{2t+1} - (1-\delta) k_{2t}) = f\left(\widetilde{k}_{2t}, l_{2t}\right)$

$\widetilde{k}_{1t} = k_{1t}, \qquad \widetilde{k}_{2t} = k_{2t}, \qquad L_t = l_{1t} + l_{2t}, \qquad M_t = (1+\mu) M_{t-1}$

We have 18 equations in 17 unknowns given the initial conditions, one is for Walras Law.

c) Simplify: $\quad r_{1t} = r_{2t} \equiv r_t \qquad u'_2(c_{1t}, c_{2t}) = p_{2t}\lambda_t \qquad v'(L_t) = w_t \lambda_t$

$\beta \lambda_{t+1} p_{2t+1}\left(\frac{r_{t+1}}{p_{2t+1}} + (1-\delta)\right) = \lambda_t p_{2t} \qquad M_{t-1} = p_{1t} c_{1t} = M_t / (1+\mu)$

$p_{1t} f'_l(k_{1t}, l_{1t}) = w_t = p_{2t} f'_l(k_{2t}, l_{2t}) \qquad p_{1t} f'_k(k_{1t}, l_{1t}) = r_t = p_{2t} f'_k(k_{2t}, l_{2t})$

$r_t(k_{1t} + k_{2t}) + w_t L_t = p_{1t} c_{1t} + p_{2t} c_{2t} + p_{2t}(k_{1t+1} - (1-\delta) k_{1t} + k_{2t+1} - (1-\delta) k_{2t})$

$c_{1t} = f(k_{1t}, l_{1t}), \qquad c_{2t} + (k_{1t+1} - (1-\delta) k_{1t} + k_{2t+1} - (1-\delta) k_{2t}) = f(k_{2t}, l_{2t}) \qquad L_t = l_{1t} + l_{2t}$

$u'_1(c_{1t}, c_{2t}) - \psi_t p_{1t} - \lambda_t p_{1t} = 0 \qquad \psi_{t+1}\beta + \lambda_{t+1}\beta(1+\mu) - \lambda_t = 0$

Steady-state: $k_1, k_2, c_1, c_2, l_1, l_2, L = const \qquad p_{2t+1}/p_{2t} = 1 + \pi \qquad M_t / p_{2t} = m$

Define : $\quad \lambda_t p_{2t} = \lambda, \qquad w_t / p_{2t} = w, \qquad p_{1t}/p_{2t} = p, \qquad r_t / p_{2t} = r, \qquad \psi_t p_{2t} = \psi$

Therefore, $\frac{M_{t-1}}{p_{2t-1}} \frac{p_{2t-1}}{p_{1t-1}} \frac{p_{1t-1}}{p_{1t}} = c_{1t} = \frac{M_t}{p_{2t}} \frac{p_{2t}}{p_{1t}} \frac{1}{(1+\mu)} \quad \Leftrightarrow \quad \frac{m}{1+\pi} = p c_1 = \frac{m}{(1+\mu)}$

$u'_1(c_1, c_2) - \psi p - \lambda p = 0 \qquad \frac{p_{2t+1}\psi_{t+1}}{p_{2t+1}}\beta + \frac{\lambda_{t+1} p_{2t+1}}{p_{2t+1}}\beta(1+\mu) - \frac{\lambda_t p_{2t}}{p_{2t}} = 0 \Leftrightarrow \frac{1}{(1+\mu)}\psi\beta + \lambda\beta = \lambda$

Therefore, $\psi = \lambda(1+\mu)\left(\frac{1}{\beta} - 1\right) \qquad u'_1(c_1, c_2) = (\psi + \lambda) p = \left[(1+\mu)\frac{1}{\beta} - \mu\right] p\lambda.$

Then the s-s is described by:

$u'_2(c_1, c_2) = \lambda \qquad u'_1(c_1, c_2) = \left[(1+\mu)\frac{1}{\beta} - \mu\right] p\lambda \qquad v'(L) = w\lambda \qquad \beta(r + (1-\delta)) = 1$

$\psi = \lambda\left(\frac{1}{\beta} - 1\right) \qquad p f'_l(k_1, l_1) = w = f'_l(k_2, l_2) \qquad p f'_k(k_1, l_1) = r = f'_k(k_2, l_2) \qquad L = l_1 + l_2$

$r(k_1 + k_2) + wL = p c_1 + c_2 + \delta k_1 + \delta k_2 \qquad c_1 = f(k_1, l_1), \qquad c_2 + \delta k_1 + \delta k_2 = f(k_2, l_2)$

$\pi = \mu \qquad p c_1 = m/(1+\mu) \qquad -$ 15 equations in 14 unknowns.

Solution:

$\lambda = u'_2 \qquad w = \frac{v'}{u'} \qquad r = \frac{1}{\beta} - (1-\delta) \qquad p = \frac{f'_k(k_2, l_2)}{f'_k(k_1, l_1)} \qquad \pi = \mu \qquad m\frac{1}{(1+\mu)} = \frac{f'_k(k_2, l_2)}{f'_k(k_{1t}, l_{1t})} c_1$

$p f'_l(k_1, l_1) = \frac{v'(l_1 + l_2)}{u'_2(c_1, c_2)} = f'_l(k_2, l_2) \qquad r = f'_k(k_2, l_2) \qquad u'_2(c_1, c_2) = u'_1(c_1, c_2)$

$c_1 = f(k_1, l_1) \qquad c_2 + \delta(k_1 + k_2) = f(k_2, l_2) \qquad u'_1 = u'_2\left[(1+\mu)\frac{1}{\beta} - \mu\right] \frac{f'_k(k_2, l_2)}{f'_k(k_1, l_1)}$

Implied from the rest and CRS: $(r - \delta)(k_1 + k_2) + \frac{v'(l_1 + l_2)}{u'_2(c_1, c_2)}(l_1 + l_2) = \frac{r}{f'_k(k_1, l_1)} c_1 + c_2.$

Now we have 6 equations in 6 unknowns - all real variables:

$\frac{f'_k(k_2, l_2)}{f'_k(k_1, l_1)} f'_l(k_1, l_1) = \frac{v'(l_1 + l_2)}{u'_2(c_1, c_2)} = f'_l(k_2, l_2) \qquad \left(\frac{1}{\beta} - (1-\delta)\right) = f'_k(k_2, l_2) \qquad c_1 = f(k_1, l_1)$

$u'_1(c_1, c_2) = u'_2(c_1, c_2)\left[(1+\mu)\frac{1}{\beta} - \mu\right] \frac{f'_k(k_2, l_2)}{f'_k(k_1, l_1)} \qquad c_2 + \delta(k_1 + k_2) = f(k_2, l_2)$

Nominal variables are defined as: $\qquad \pi = \mu \qquad m\frac{1}{(1+\mu)} = \frac{f'_k(k_2, l_2)}{f'_k(k_{1t}, l_{1t})} c_1.$

(d) Now the equation, connecting the two marginal utilities depends on μ. Therefore inflation changes propagate through the whole system, changing all the values. It also follows that money growth is still directly reflected in the price level: $\pi = \mu$. Money is neutral but not superneutral here. The CIA requirement introduces a wedge between marginal utilities of the two types of goods. It is exactly as if consumer faced a tax of $\left[(1+\mu)\frac{1}{\beta} - \mu\right]\beta - 1 = \mu - \mu\beta$. Non-distorting situation is the optimal one. The economy shrinks as the tax increases.

The optimal level of money growth should satisfy: $\quad \left[(1+\mu)\frac{1}{\beta} - \mu\right] = \frac{1}{\beta} \quad \Leftrightarrow \quad \mu = 0$

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Exercise 3 *Static Incomplete Information Problem*

Uncertainty: $\overline{\alpha}\underline{M} = \underline{\alpha}\overline{M} \qquad (1-\overline{\alpha})\overline{M} = \underline{\alpha}\overline{M} = \overline{\alpha}\underline{M} = (1-\underline{\alpha})\underline{M}$

$$(\alpha, M) = \begin{cases} (\overline{\alpha}, \overline{M}) & 1/4 \\ (\overline{\alpha}, \underline{M}) & 1/4 \\ (\underline{\alpha}, \overline{M}) & 1/4 \\ (\underline{\alpha}, \underline{M}) & 1/4 \end{cases}, \alpha M = \begin{cases} \overline{\alpha}\overline{M} & 1/4 \\ \overline{\alpha}\underline{M} & 1/4 \\ \overline{\alpha}\underline{M} & 1/4 \\ \underline{\alpha}\underline{M} & 1/4 \end{cases}, \qquad (1-\alpha) M = \begin{cases} \overline{\alpha}\underline{M} & 1/4 \\ \underline{\alpha}\underline{M} & 1/4 \\ \overline{\overline{\alpha}\overline{M}} & 1/4 \\ \overline{\alpha}\underline{M} & 1/4 \end{cases}$$

Demands: $Q_A = \frac{\alpha M}{p_A} \qquad Q_B = \frac{(1-\alpha)M}{p_B}$ Profits: $\pi_i = \frac{p_i q_i}{M} - \frac{1}{2}q_i^2.$ Measure 1.

(a) In the complete information case they see the pair (α, M) and find the optimal price and quantity: A) $\pi_A|\alpha, M = \frac{p_A q_A}{M} - \frac{1}{2}q_A^2|p_A, \alpha, M \to \max_{q_A}$ Market clearing: $q_A = Q_A = \frac{\alpha M}{p_A}$

FOC: $q_A = \frac{p_A}{M}$ Market clearing: $\frac{\alpha M}{p_A} = \frac{p_A}{M}$

Equilibrium: $p_A = M\sqrt{\alpha} \qquad q_A = \sqrt{\alpha} \qquad \pi_A = \alpha/2$

B) $\pi_B|\alpha, M = \frac{p_B q_B}{M} - \frac{1}{2}q_B^2|p_B, \alpha, M \to \max_{q_B}$ Market clearing: $q_B = Q_B = \frac{(1-\alpha)M}{p_B}$

FOC: $q_B = \frac{p_B}{M}$ Market clearing: $\frac{(1-\alpha)M}{p_B} = \frac{pB}{M}$

Equilibrium: $p_A = M\sqrt{1-\alpha} \qquad q_A = \sqrt{1-\alpha} \qquad \pi_A = (1-\alpha)/2$

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(b) $E\pi_A = E\left[\frac{p_A q_A}{M} - \frac{1}{2}q_A^2|p_A\right] \to \max_{q_A}$ Market clearing: $q_A = Q_A = \frac{\alpha M}{p_A}$

$E\pi_B = E\left[\frac{p_A q_A}{M} - \frac{1}{2}q_A^2|p_A\right] \to \max_{q_A}$ Market clearing: $q_A = Q_A = \frac{\alpha M}{p_A}$

Equilibrium: Agents form beliefs on α and M given the price. They behave optimally given their beliefs. Beliefs are bayesian and agents correctly forecast statistical properties of endogenous observables. Markets clear.

(c) Guess that the price reveals the product $M\alpha$.

The products can take 3 values. For example, for A $\alpha M = \begin{cases} \overline{\alpha}\overline{M} & 1/4 \\ \overline{\alpha}\underline{M} = \underline{\alpha}\overline{M} & 1/2 \\ \underline{\alpha}\underline{M} & 1/4 \end{cases}$

In two of the cases he can figure out both parameters knowing the product, in the third case there are two equally possible outcomes.

$$E[\pi_A|\alpha M] = \left\{ \begin{array}{c|c} \frac{p_A q_A}{\overline{M}} - \frac{1}{2}q_A^2 & \overline{\alpha}\overline{M} \\ \frac{1}{2}\left(\frac{1}{\overline{M}} + \frac{1}{\underline{M}}\right) p_A q_A - \frac{1}{2}q_A^2 & \overline{\alpha}\underline{M} \\ \frac{p_A q_A}{\underline{M}} - \frac{1}{2}q_A^2 & \underline{\alpha}\underline{M} \end{array} \right. \qquad \Rightarrow \qquad q_A|\alpha M = \left\{ \begin{array}{c|c} \frac{p_A}{\overline{M}} & \overline{\alpha}\overline{M} \\ \frac{1}{2}\left(\frac{p_A}{\overline{M}} + \frac{p_A}{\underline{M}}\right) & \overline{\alpha}\underline{M} \\ \frac{p_A}{\underline{M}} & \underline{\alpha}\underline{M} \end{array} \right.$$

$$q_A = \left\{ \begin{array}{c|c} \frac{p_A}{\overline{M}} & \overline{\alpha}\overline{M} \\ \frac{1}{2}\left(\frac{p_A}{\overline{M}} + \frac{p_A}{\underline{M}}\right) & \overline{\alpha}\underline{M} \\ \frac{1}{2}\left(\frac{p_A}{\overline{M}} + \frac{p_A}{\underline{M}}\right) & \underline{\alpha}\overline{M} \\ \frac{p_A}{\underline{M}} & \underline{\alpha}\underline{M} \end{array} \right. = \left\{ \begin{array}{c|c} \frac{\overline{\alpha}\overline{M}}{p_A} & \overline{\alpha}\overline{M} \\ \frac{\overline{\alpha}\underline{M}}{p_A} & \overline{\alpha}\underline{M} \\ \frac{\underline{\alpha}\overline{M}}{p_A} & \underline{\alpha}\overline{M} \\ \frac{\underline{\alpha}\underline{M}}{p_A} & \underline{\alpha}\underline{M} \end{array} \right. = Q_A$$

$$p_A = \left\{ \begin{array}{c|c} \overline{M}\sqrt{\overline{\alpha}} & \overline{\alpha}\overline{M} \\ \sqrt{2\overline{\alpha}\underline{M}/\left(\frac{1}{\overline{M}} + \frac{1}{\underline{M}}\right)} & \overline{\alpha}\underline{M} \\ \sqrt{2\underline{\alpha}\overline{M}/\left(\frac{1}{\overline{M}} + \frac{1}{\underline{M}}\right)} & \underline{\alpha}\overline{M} \\ \underline{M}\sqrt{\underline{\alpha}} & \underline{\alpha}\underline{M} \end{array} \right. \qquad q_A = \left\{ \begin{array}{c|c} \sqrt{\overline{\alpha}} & \overline{\alpha}\overline{M} \\ \sqrt{\overline{\alpha}\underline{M}\left(\frac{1}{\overline{M}} + \frac{1}{\underline{M}}\right)/2} & \overline{\alpha}\underline{M} \\ \sqrt{\underline{\alpha}\overline{M}\left(\frac{1}{\overline{M}} + \frac{1}{\underline{M}}\right)/2} & \underline{\alpha}\overline{M} \\ \sqrt{\underline{\alpha}} & \underline{\alpha}\underline{M} \end{array} \right.$$

We have confirmed, that he can distinguish two cases of three given the price, and cannot distinguish the other two between themselves. We have verified our guess.

Same for B: $\quad (1-\alpha)M = \begin{cases} \overline{\alpha}\underline{M} & 1/4 \leftarrow \\ \underline{\alpha}\underline{\underline{M}} & 1/4 \\ \overline{\alpha}\overline{\overline{M}} & 1/4 \\ \underline{\alpha}\overline{M} & 1/4 \leftarrow \end{cases}$

The agents appear to be exactly symmetric, only will distinguish different combinations.

$$E[\pi_B|\alpha M] = \left\{ \begin{array}{c|c} \frac{p_B q_B}{\overline{M}} - \frac{1}{2}q_B^2 & \overline{\alpha}\overline{M} \\ \frac{1}{2}\left(\frac{1}{\overline{M}} + \frac{1}{\underline{M}}\right) p_B q_B - \frac{1}{2}q_B^2 & \overline{\alpha}\underline{M} \\ \frac{p_B q_B}{\underline{M}} - \frac{1}{2}q_B^2 & \underline{\alpha M} \end{array} \right. \Rightarrow \quad q_B|\alpha M = \left\{ \begin{array}{c|c} \frac{p_B}{\overline{M}} & \overline{\alpha}\overline{M} \\ \frac{1}{2}\left(\frac{p_B}{\overline{M}} + \frac{p_B}{\underline{M}}\right) & \overline{\alpha}\underline{M} \\ \frac{p_B}{\underline{M}} & \underline{\alpha M} \end{array} \right.$$

$$q_B = \left\{ \begin{array}{c|c} \frac{p_B}{\overline{M}} & \overline{\alpha}\overline{M} \\ \frac{1}{2}\left(\frac{p_B}{\overline{M}} + \frac{p_B}{\underline{M}}\right) & \overline{\alpha}\underline{M} \\ \frac{1}{2}\left(\frac{p_B}{\overline{M}} + \frac{p_B}{\underline{M}}\right) & \underline{\alpha}\overline{M} \\ \frac{p_B}{\underline{M}} & \underline{\alpha M} \end{array} \right. = \left\{ \begin{array}{c|c} \frac{\overline{\alpha}\overline{M}}{p_B} & \overline{\alpha}\overline{M} \\ \frac{\overline{\alpha}\underline{M}}{p_B} & \overline{\alpha}\underline{M} \\ \frac{\underline{\alpha}\overline{M}}{p_B} & \underline{\alpha}\overline{M} \\ \frac{\underline{\alpha M}}{p_B} & \underline{\alpha M} \end{array} \right. = Q_B$$

$$p_B = \left\{ \begin{array}{c|c} \overline{M}\sqrt{\overline{\alpha}} & \overline{\alpha}\overline{M} \\ \sqrt{2\overline{\alpha}\underline{M}/\left(\frac{1}{\overline{M}} + \frac{1}{\underline{M}}\right)} & \overline{\alpha}\underline{M} \\ \sqrt{2\underline{\alpha}\overline{M}/\left(\frac{1}{\overline{M}} + \frac{1}{\underline{M}}\right)} & \underline{\alpha}\overline{M} \\ \underline{M}\sqrt{\underline{\alpha}} & \underline{\alpha M} \end{array} \right. \qquad q_B = \left\{ \begin{array}{c|c} \sqrt{\overline{\alpha}} & \overline{\alpha}\overline{M} \\ \sqrt{\overline{\alpha}\underline{M}\left(\frac{1}{\overline{M}} + \frac{1}{\underline{M}}\right)/2} & \overline{\alpha}\underline{M} \\ \sqrt{\underline{\alpha}\overline{M}\left(\frac{1}{\overline{M}} + \frac{1}{\underline{M}}\right)/2} & \underline{\alpha}\overline{M} \\ \sqrt{\underline{\alpha}} & \underline{\alpha M} \end{array} \right.$$

We have confirmed, that he can distinguish two cases of three given the price, and cannot distinguish the other two between themselves. We have verified our guess for B as well.

(d) We can see, that in half of the situations the outcomes are optimal. To compare welfare we shall compare expected profits of the economy:

$\pi_j = \frac{1}{2}\left(\frac{1}{\overline{M}} + \frac{1}{\underline{M}}\right)\overline{\alpha}\underline{M} - \frac{1}{4}\overline{\alpha}\underline{M}\left(\frac{1}{\overline{M}} + \frac{1}{\underline{M}}\right) = \frac{1}{4}\overline{\alpha}\underline{M}\left(\frac{1}{\overline{M}} + \frac{1}{\underline{M}}\right) = \frac{1}{4}\overline{\alpha}\left(\frac{\underline{\alpha}}{\overline{\alpha}} + 1\right) = \frac{\underline{\alpha}+\overline{\alpha}}{4} = \frac{1}{4}$

Optimum: $\quad \pi_A + \pi_B = \frac{1}{2}\alpha + \frac{1}{2}(1-\alpha) = \frac{1}{2}$

Second-best: $\quad E\pi_A + E\pi_B = \frac{1}{4}\alpha + \frac{1}{4}(1-\alpha) + \frac{1}{2}\pi_j = \frac{3}{8} < \frac{1}{2}$

Very similar to what we saw in class. No full information => no efficiency. But even some information helps.

Exercise 4 *One-time Money Injection*

(a) Final goods' sector: $\quad \max_{y_t} P_t Y_t - \int_0^1 p_t^i y_t^i di \qquad$ s.t. $Y_t = \left[\int_0^1 (y_t^i)^{\frac{\theta-1}{\theta}} di\right]^{\frac{\theta}{\theta-1}}$

FOC: $\quad P_t \frac{\theta}{\theta-1}\frac{\theta-1}{\theta}(y_t^i)^{\frac{\theta-1}{\theta}-1}\left[\int_0^1 (y_t^i)^{\frac{\theta-1}{\theta}} di\right]^{\frac{\theta}{\theta-1}-1} = p_t^i$

$P_t (y_t^i)^{\frac{-1}{\theta}}\left[\int_0^1 (y_t^i)^{\frac{\theta-1}{\theta}} di\right]^{\frac{1}{\theta-1}} = p_t^i \qquad \boxed{y_t^i} = \left(\frac{P_t}{p_t^i}\right)^\theta \left[\int_0^1 (y_t^i)^{\frac{\theta-1}{\theta}} di\right]^{\frac{\theta}{\theta-1}} = \boxed{Y_t\left(\frac{P_t}{p_t^i}\right)^\theta}$

$P_t Y_t = \int_0^1 p_t^i y_t^i di = \int_0^1 p_t^i Y_t \left(\frac{P_t}{p_t^i}\right)^\theta di = Y_t P_t^\theta \int_0^1 (p_t^i)^{1-\theta} di \qquad \boxed{P_t = \left[\int_0^1 (p_t^i)^{1-\theta} di\right]^{\frac{1}{1-\theta}}}$

Intermediate good's sector:

$\pi_t^i = p_t^i y_t^i - M_t n_t^i \to \max_{n_t^i, p_t^i, y_t^i} \qquad$ s.t. $\quad y_t^i = (n_t^i)^\alpha \qquad$ s.t. $Y_t\left(\frac{P_t}{p_t^i}\right)^\theta = y_t^i$

$\pi_t^i = p_t^i Y_t \left(\frac{P_t}{p_t^i}\right)^\theta - M_t\left(Y_t\left(\frac{P_t}{p_t^i}\right)^\theta\right)^{\frac{1}{\alpha}} \to \max_{p_t^i}$

(b) Flexible prices: FOC: $(1-\theta) Y_t \left(\frac{P_t}{p_t^i}\right)^{\theta} + \frac{\theta}{\alpha}\frac{M_t}{p_t^i}\left(Y_t \left(\frac{P_t}{p_t^i}\right)^{\theta}\right)^{\frac{1}{\alpha}} = 0$

Therefore, $(\theta-1)(P_t Y_t)^{1-\frac{1}{\alpha}} P_t^{\frac{1}{\alpha}} \left(\frac{p_t^i}{P_t}\right)^{1+\frac{\theta}{\alpha}-\theta} = \frac{\theta}{\alpha} M_t$

Household: $P_t Y_t = M_t$

Equilibrium: $(\theta-1) M_t^{1-\frac{1}{\alpha}} P_t^{\frac{1}{\alpha}} \left(\frac{p_t^i}{P_t}\right)^{1+\frac{\theta}{\alpha}-\theta} = \frac{\theta}{\alpha} M_t$

Hence, $\left(\frac{p_t^i}{P_t}\right)^{1+\frac{\theta}{\alpha}-\theta} = \frac{1}{\alpha}\frac{\theta}{\theta-1}\left(\frac{M_t}{P_t}\right)^{\frac{1}{\alpha}}, \qquad P_t = \left[\int_0^1 (p_t^i)^{1-\theta} di\right]^{\frac{1}{1-\theta}}.$

All agents are all the same and fully adjust: $p_t^i = P_t = M_t \left(\frac{1}{\alpha}\frac{\theta}{\theta-1}\right)^{\frac{1}{\alpha}}$

(c) Calvo-sticky prices: can only adjust from time to time $\Rightarrow$ losses.

Oprimal price, if could adjust: $p_t^{i*} = P_t = M_t \left(\frac{1}{\alpha}\frac{\theta}{\theta-1}\right)^{\frac{1}{\alpha}}$

Derive: if cannot adjust: $\pi_t^i = p_t^i Y_t \left(\frac{P_t}{p_t^i}\right)^{\theta} - M_t \left(Y_t\left(\frac{P_t}{p_t^i}\right)^{\theta}\right)^{\frac{1}{\alpha}} =$

$= p_t^i \frac{M_t}{P_t}\left(\frac{P_t}{p_t^i}\right)^{\theta} - M_t \left(Y_t\left(\frac{P_t}{p_t^i}\right)^{\theta}\right)^{\frac{1}{\alpha}} = M_t \left(\frac{P_t}{p_t^i}\right)^{\theta-1} - \frac{M_t}{\frac{1}{\alpha}\frac{\theta}{\theta-1}}\left(\frac{P_t}{p_t^i}\right)^{\frac{\theta}{\alpha}} = M_t\left(x^{\theta-1} - a x^{\theta/\alpha}\right)$

$x = \frac{P_t}{p_t^i} \qquad x^* = \frac{P_t}{p_t^{i*}} \qquad x - x^* = \frac{p_t^{i*}}{p_t^i} - 1 \approx (\log p_t^* - \log p_t^i)$

$\pi_t^i(x) - \pi_t^i(x^*) = M_t \frac{\partial}{\partial x}\left(x^{\theta-1} - a x^{\theta/\alpha}\right)\Big|_{x=x^*=1}(x-x^*) + \frac{M_t}{2}\frac{\partial^2}{\partial x^2}\left(x^{\theta-1} - a x^{\theta/\alpha}\right)\Big|_{x=x^*=1}(x-x^*)^2 =$

$= M_t \frac{\partial}{\partial x}\left(x^{\theta-1} - a x^{\theta/\alpha}\right)\Big|_{x=x^*=1}(x-x^*) + \frac{M_t}{2}\frac{\partial^2}{\partial x^2}\left(x^{\theta-1} - a x^{\theta/\alpha}\right)\Big|_{x=x^*=1}(x-x^*)^2$

$\frac{\partial}{\partial x}\left(x^{\theta-1} - a x^{\theta/\alpha}\right)\Big|_{x=x^*=1} = (\theta-1)x^{\theta-2} - \frac{a}{x}\frac{\theta}{\alpha}x^{\frac{\theta}{\alpha}}\Big|_{x=x^*=1} = (\theta-1) - \frac{1}{\frac{1}{\alpha}\frac{\theta}{\theta-1}}\frac{\theta}{\alpha} = 0$

$-g = \frac{\partial^2}{\partial x^2}\left(x^{\theta-1} - a x^{\theta/\alpha}\right)\Big|_{x=x^*=1} = \left(\theta - \frac{\theta}{\alpha} - 1\right)(\theta-1) < 0$ for $\alpha < 1$.

Therefore profits are approximately equal to $\pi_t^i(x) = \pi_t^i(x^*) - \frac{M_t}{2} g (\ln p_t^* - \ln p_t^i)^2$

Intermediate firm's problem: $E\sum_{t=0}^{\infty}\beta^t (\ln p_t^* - \ln p_t^i)^2 \to \min_{p_t^i}$

If only fraction λ can adjust, then the probability of not being able to adjust t consecutive periods is $(1-\lambda)^t$. The price is fixed before adjusting, and the loss is zero afterwards. Hence,

$E\sum_{t=0}^{\infty}\beta^t (\ln p_t^* - \ln p_t^i)^2 = \sum_{t=0}^{\infty}\beta^t (1-\lambda)^t E(\ln p_t^* - \ln p_t^i)^2 \to \min_{p_t^i}$

FOC: $\sum_{t=0}^{\infty}\beta^t(1-\lambda)^t E(\ln p_t^* - \ln p_t^i) = 0 \qquad \ln p_t^i = (\beta\lambda - \beta + 1)\sum_{j=0}^{\infty}\beta^j (1-\lambda)^j E_0 \ln p_{t+j}^*$

$\ln p_t^i = (\beta\lambda-\beta+1)\ln p_t^* + (\beta\lambda-\beta+1)\beta(1-\lambda)\sum_{j=1}^{\infty}\beta^{j-1}(1-\lambda)^{j-1} E\ln p_{t+j}^*$

$\boxed{\ln p_t^i = (\beta\lambda-\beta+1)\ln p_t^* + \beta(1-\lambda)E\ln p_{t+1}^i}$

Besides, since fraction λ adjusts, we have that $\boxed{\ln P_t = (1-\lambda)\ln P_{t-1} + \lambda \ln p_t^i}$

Therefore, $[\ln p_t^i - \ln P_t] = \frac{(1-\lambda)}{\lambda}[\ln P_t - \ln P_{t-1}] = \frac{(1-\lambda)}{\lambda}\pi_t$

Substract a constant: $[\ln p_t^i - \ln P_t] = (\beta\lambda-\beta+1)[\ln p_t^* - \ln P_t] + \beta(1-\lambda)E\left[\ln p_{t+1}^i - \ln P_t\right]$

$\pi_t = \frac{\lambda}{1-\lambda}(\beta\lambda-\beta+1)[\ln p_t^* - \ln P_t] + \beta(1-\lambda)E\pi_{t+1}$

From $\left(\frac{p_t^*}{P_t}\right)^{1+\frac{\theta}{\alpha}-\theta} = \frac{1}{\alpha}\frac{\theta}{\theta-1}\left(\frac{M_t}{P_t}\right)^{\frac{1}{\alpha}}$, it follows that

$\ln p_t^* = \frac{1}{\left(1+\frac{\theta}{\alpha}-\theta\right)}\ln\frac{1}{\alpha}\frac{\theta}{\theta-1} + \frac{\frac{1}{\alpha}}{1+\frac{\theta}{\alpha}-\theta}\ln M_t + \left(1 - \frac{\frac{1}{\alpha}}{1+\frac{\theta}{\alpha}-\theta}\right)\ln P_t$

$\boxed{[\ln p_t^* - \ln P_t] = \frac{1}{\left(1+\frac{\theta}{\alpha}-\theta\right)}\ln\frac{1}{\alpha}\frac{\theta}{\theta-1} + \frac{\frac{1}{\alpha}}{1+\frac{\theta}{\alpha}-\theta}[\ln M_t - \ln P_t]} = K + \varphi[\ln M_t - \ln P_t]$ Therefore,

Finally, $\ln P_t - \ln P_{t-1} = \frac{\lambda}{1-\lambda}(\beta\lambda-\beta+1)\left(\frac{1}{\left(1+\frac{\theta}{\alpha}-\theta\right)}\ln\frac{1}{\alpha}\frac{\theta}{\theta-1} + \frac{\frac{1}{\alpha}}{1+\frac{\theta}{\alpha}-\theta}[\ln M_t - \ln P_t]\right) +$

$+\beta(1-\lambda)E\ln[P_{t+1}-\ln P_t]$ Solving recursively under perfect foresight we get:
$\ln P_t - \ln P_{t-1} = \gamma(K+\varphi[\ln M_t - \ln P_t]) + \delta E[\ln P_{t+1} - \ln P_t] = \gamma\sum_{j=0}^{\infty}\delta^j E(K+\varphi[\ln M_t - \ln P_{t+j}])$

(d) If there is no uncertainty (M=const) and we are in the flexible price equilibrium:
$[\ln p_t^* - \ln P_t] = K + \varphi[\ln M - \ln P_t]$ $[\ln p_t^i - \ln P_t] = \frac{(1-\lambda)}{\lambda}[\ln P_t - \ln P_{t-1}]$ $p_t^i = p_t^*$
Then, $K + \varphi[\ln M - \ln P_t] = \frac{(1-\lambda)}{\lambda}[\ln P_t - \ln P_{t-1}]$
$\frac{(1-\lambda)}{\lambda}\left\{\sum_{j=0}^{\infty}\delta^j\right\}[\ln P_t - \ln P_{t-1}] = \sum_{j=0}^{\infty}\delta^j E(K+\varphi[\ln M - \ln P_{t+j}])$
We check, that $K + \varphi[\ln M - \ln P_{t+j}] = \frac{(1-\lambda)}{\lambda}[\ln P_{t+j} - \ln P_{t+j-1}]$ satisfies this equation.
This implies that prices remain constant $\ln P_{t-1} = \ln P_t = \ln P_{t+1} = ... = \frac{K}{\varphi} + \ln M$.
Therefore, staying in the flexible price equilibrium is a solution.

(e) Now, assume everybody knows at time 0 that at time T money stock M jumps to M'.
Then the following system of equations must hold:
$\ln P_t - \ln P_{t-1} = \frac{\lambda}{1-\lambda}(\beta\lambda - \beta + 1)\frac{\frac{1}{\alpha}}{1+\frac{\theta}{\alpha}-\theta}\left(\alpha\ln\frac{1}{\alpha}\frac{\theta}{\theta-1} + \ln M_t - \ln P_t\right) + \beta(1-\lambda)[\ln P_{t+1} - \ln P_t]$
$M_t = \begin{cases} M & t = 0,..T-1 \\ M' & t = T,....\infty \end{cases}$ given $P_{-1} = \alpha\ln\frac{1}{\alpha}\frac{\theta}{\theta-1} + \ln M_t$
Proof of convergence (more general):
Equations could be rewritten as: $\ln\frac{P_t}{P_{t-1}} = \gamma\varphi\ln\frac{P_t^*}{P_t} + \delta E\ln\frac{P_{t+1}}{P_t} = \gamma\varphi\sum_{j=0}^{\infty}\delta^j E\ln\frac{P_{t+j}^*}{P_{t+j}}$
Guess, $\ln\frac{P_t}{P_{t-1}} = \psi\ln\frac{P_t^*}{P_t}$ $\Rightarrow$ $\ln\frac{P_{t+1}}{P_t} = \psi\ln\frac{P_{t+1}^*}{P_{t+1}}$
$\Rightarrow$ $\psi\ln\frac{P_t^*}{P_t} = \gamma\varphi\ln\frac{P_t^*}{P_t} + \delta E\ln\frac{P_{t+1}}{P_t} = \gamma\varphi\ln\frac{P_t^*}{P_t} + \delta\psi E\ln\frac{P_{t+1}^*}{P_{t+1}}$
Under perfect foresigh we have: $\frac{(\psi-\gamma\varphi)}{\delta\psi}\ln\frac{P_t^*}{P_t} = E\ln\frac{P_{t+1}^*}{P_{t+1}} = \ln\frac{P_t^*}{P_{t+1}}$ $\Leftrightarrow$
$\left(\frac{(\psi-\gamma\varphi)}{\delta\psi} - 1\right)\ln\frac{P_t^*}{P_t} = \ln\frac{P_t}{P_{t+1}}$ We have veryfied the guess. Should be $\left(\frac{(\psi-\gamma\varphi)}{\delta\psi} - 1\right) = \psi$.
This leads to a quadratic equation: $\psi - \gamma\varphi - \delta\psi = \delta\psi^2$, Solution is:
$\psi = \frac{1}{2}\left(\sqrt{1 + \frac{1}{\delta^2} - 2\frac{1}{\delta}(1-2\varphi\gamma)} + \frac{1}{\delta} - 1\right)$ $\delta = \beta(1-\lambda)$ $\gamma\varphi = \frac{\lambda}{1-\lambda}\frac{1-\beta(1-\lambda)}{\theta-\alpha(\theta-1)}$
For $D = 1 + \frac{1}{\delta^2} - 2\frac{1}{\delta}(1-2\varphi\gamma) = \frac{4}{\delta}\varphi\gamma + \left(\frac{1}{\delta} - 1\right)^2 > 0$ need $\frac{\theta}{\theta-1} > \alpha, 0 < \lambda < \frac{1}{\beta}$
Values such that $0 < \lambda < 1, 0 < \beta < 1, 0 < \alpha < 1$ and $\theta > 1$ would work. Price converges to P^*.
Define $\ln Y_t = \alpha\ln\frac{1}{\alpha}\frac{\theta}{\theta-1} + \ln M_t$. Could be simplified and solved numerically:
$0 = \gamma\varphi\ln Y_t + \ln P_{t-1} - (1+\delta+\gamma\varphi)\ln P_t + \delta\ln P_{t+1}$
An example of a transition path for: $\alpha = 1/2, \beta = 0.9, \lambda = 0.4, \theta = 3/2, M' = 2M, T = 20$
We see the same result as in class: speed of convergenc grows as α and λ grow and θ falls.