Econ202C Homework #1 Anton Cheremukhin

April 21, 2006

**Exercise 1**

$\max E\Sigma_{t=0}^{\infty}\beta^t\alpha_t\frac{C_t^{1-\sigma}-1}{1-\sigma}$ $\quad X_t=\gamma^t X_0=\gamma^t$ $\quad N_t=\varkappa^t N_0=\varkappa^t$

$AK_t^{\theta}\left(X_tN_t\right)^{1-\theta}+\left(1-\delta\right)K_t\geq C_t+K_{t+1}$ $\quad \alpha\in\left\{\frac{3}{4},1,\frac{5}{4}\right\}$

$$P_4=\begin{bmatrix} .15 & .7 & .15\\ .10 & .8 & .10\\ .15 & .7 & .15\end{bmatrix}\qquad P_5=\begin{bmatrix} .8 & .1 & .1\\ .1 & .8 & .1\\ .1 & .1 & .8\end{bmatrix}$$

**1)** Use L'Hopital rule:

$$\lim_{\sigma\to1}\frac{C^{1-\sigma}-1}{1-\sigma}=\lim_{\sigma\to1}\frac{\frac{\partial}{\partial\sigma}\left[C^{1-\sigma}-1\right]}{\frac{\partial}{\partial\sigma}[1-\sigma]}=\lim_{\sigma\to1}\frac{\frac{\partial}{\partial\sigma}\left[e^{(1-\sigma)\log C}\right]}{-1}=\lim_{\sigma\to1}\frac{\left[e^{(1-\sigma)\log C}\right](-\log C)}{-1}=\log C$$

Transform everything using the fact that $X_tN_t=(\gamma\varkappa)^t$:

$$A\left(\frac{K_t}{X_tN_t}\right)^{\theta}+\left(1-\delta\right)\frac{K_t}{X_tN_t}\geq\frac{C_t}{X_tN_t}+\frac{K_{t+1}}{X_{t+1}N_{t+1}}\frac{X_{t+1}N_{t+1}}{X_tN_t}=\frac{C_t}{X_tN_t}+\frac{K_{t+1}}{X_{t+1}N_{t+1}}\left(\gamma\varkappa\right)$$

define $\frac{K_t}{X_tN_t}=k_t$ $\quad \frac{C_t}{X_tN_t}=c_t$ $\quad\Rightarrow\quad Ak_t{}^{\theta}+\left(1-\delta\right)k_t\geq c_t+\gamma\varkappa k_{t+1}$

$$E\Sigma_{t=0}^{\infty}\beta^t\alpha_t\frac{C_t^{1-\sigma}-1}{1-\sigma}=E\Sigma_{t=0}^{\infty}\beta^t\alpha_t\frac{\left(X_tN_tc_t\right)^{1-\sigma}-1}{1-\sigma}=E\Sigma_{t=0}^{\infty}\left(\beta\gamma\varkappa\right)^t\alpha_t\frac{\left(c_t\right)^{1-\sigma}-1}{1-\sigma}+const$$

Problem restated as: $\max E\Sigma_{t=0}^{\infty}\left[\left(\beta\gamma\varkappa\right)^t\alpha_t\right]\frac{\left(c_t\right)^{1-\sigma}-1}{1-\sigma}$ $\quad$ s.t. $Ak_t{}^{\theta}+\left(1-\delta\right)k_t\geq c_t+\gamma\varkappa k_{t+1}$

Recursive formulation: $v\left(k,s\right)=\max_{k'}\left[\alpha\left(s\right)\frac{\left(Ak^{\theta}+(1-\delta)k-\gamma\varkappa k'\right)^{1-\sigma}-1}{1-\sigma}+\left(\beta\gamma\varkappa\right)E_{s'|s}v\left(k',s'\right)\right]$

FOC: $\quad\alpha\left(s\right)\frac{\left(Ak^{\theta}+(1-\delta)k-\gamma\varkappa k'\right)^{-\sigma}}{1-\sigma}\left(\gamma\varkappa\right)\left(1-\sigma\right)=\left(\beta\gamma\varkappa\right)E_{s'|s}v_k'\left(k',s'\right)$

ENV: $\quad v_k'\left(k,s\right)=\alpha\left(s\right)\frac{\left(Ak^{\theta}+(1-\delta)k-\gamma\varkappa k'\right)^{-\sigma}}{1-\sigma}\left(1-\sigma\right)\left[A\theta k^{\theta-1}+\left(1-\delta\right)\right]$

Steady state: $k=k'$, $\quad\alpha_{s^*}=1$ $\quad E_{s'|s}v_k'\left(k',s'\right)=v_k'\left(k,s\right)$

$$\alpha\left(s\right)\frac{\left(Ak^{\theta}+(1-\delta)k-\gamma\varkappa k'\right)^{-\sigma}}{1-\sigma}\left(\gamma\varkappa\right)\left(1-\sigma\right)=\alpha\left(s\right)\left(\beta\gamma\varkappa\right)\frac{\left(Ak^{\theta}+(1-\delta)k-\gamma\varkappa k'\right)^{-\sigma}}{1-\sigma}\left(1-\sigma\right)\left[A\theta k^{\theta-1}+\left(1-\delta\right)\right]$$

$$1=\beta\left[A\theta k^{\theta-1}+\left(1-\delta\right)\right]\quad\Rightarrow\quad k^*=\left[\frac{\frac{1}{\beta}-(1-\delta)}{A\theta}\right]^{\frac{1}{\theta-1}},\qquad Ak^{\theta}=\frac{\frac{1}{\beta}-(1-\delta)}{\theta}k$$

$$c^*=Ak^{\theta}+\left(1-\delta\right)k-\gamma\varkappa k=\left[\frac{\frac{1}{\beta}-(1-\delta)}{\theta}-\left(\gamma\varkappa-\left(1-\delta\right)\right)\right]\left[\frac{\frac{1}{\beta}-(1-\delta)}{A\theta}\right]^{\frac{1}{\theta-1}}$$

**2)** We have values for $\gamma=1.02$, $\varkappa=1.015$, $\sigma=1$.

Firm problem (F): $\quad AK_t^{\theta}\left(X_tN_t\right)^{1-\theta}-r_tK_t-w_tN_t\to\max_{K_t,N_t}$

$\theta A\frac{K}{XN}^{\theta-1}=\theta Ak^{\theta-1}=r$ $\quad\left(1-\theta\right)AK^{\theta}\left(XN\right)^{1-\theta}\frac{1}{N}=\left(1-\theta\right)Ak^{\theta}X=w$

Ratio of output paid to labor: $\quad\frac{wN}{wN+rK}=\frac{(1-\theta)Ak^{\theta}}{\theta Ak^{\theta-1}\frac{K}{XN}+(1-\theta)Ak^{\theta}}=\left(1-\theta\right)=\frac{2}{3}$ $\quad$ I.e. $\theta=\frac{1}{3}$

Steady state investment/output ratio: $\quad c^*=\left[\frac{\frac{1}{\beta}-(1-\delta)}{\theta}-\left(\gamma\varkappa-\left(1-\delta\right)\right)\right]\left[\frac{(\gamma\varkappa)-(1-\delta)}{A\theta}\right]^{\frac{1}{\theta-1}}$

$$\frac{I_t}{Y_t}=1-\frac{C_t}{AK_t^{\theta}(X_tN_t)^{1-\theta}}=1-\frac{c^*}{Ak^{*\theta}}=1-\frac{\left[\frac{\frac{1}{\beta}-(1-\delta)}{\theta}-(\gamma\varkappa-(1-\delta))\right]}{\frac{\frac{1}{\beta}-(1-\delta)}{\theta}}=\theta\left[\frac{\gamma\varkappa-(1-\delta)}{\frac{1}{\beta}-(1-\delta)}\right]=0.25$$

Steady-state per capita capital stock: $\quad k^*=\left[\frac{\frac{1}{\beta}-(1-\delta)}{A\theta}\right]^{\frac{1}{\theta-1}}=1$

.

Steady state real return to capital (net of depreciation):

$r-\delta=\theta Ak^{\theta-1}-\delta=\left[\frac{1}{\beta}-(1-\delta)\right]k^{*}-\delta=\frac{1}{\beta}-1=0.08$ Hence, $\beta=\frac{25}{27}$

$\frac{1}{3}\left[\frac{1\frac{2}{100}*1\frac{3}{200}-(1-\delta)}{\frac{27}{25}-(1-\delta)}\right]=\frac{1}{4}\Leftrightarrow\delta=\frac{247}{2500}$ $\frac{\frac{27}{25}-1+\frac{247}{2500}}{\frac{1}{3}A}=1\Leftrightarrow A=\frac{1341}{2500}$

**3)** To decentralize the economy we need to state the household's problem and resourse constraint:

(HH): $E\Sigma_{t=0}^{\infty}\beta^{t}\alpha_{t}\frac{C_t^{1-\sigma}-1}{1-\sigma}\rightarrow\max_{C_t,K_{t+1}}$ s.t. $C_t+K_{t+1}\leq r_tK_t+(1-\delta)K_t+w_tN_t$

(RC): $AK_t^{\theta}(X_tN_t)^{1-\theta}+(1-\delta)K_t\geq C_t+K_{t+1}$ $K_1$ given

Sequential competitive equilibrium is the sequence of allocations $\{K_t,C_t,N_t\}$ and prices $\{r_t,w_t\}$:

1) The allocation is the solution of both (F) and (HH) given prices in each period

2) The allocations satisfy the resource constraint.

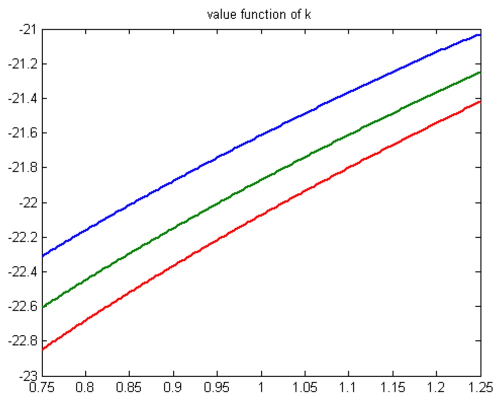
Bellman equation: $v(k,s)=\max_{k'}\left[\alpha(s)\frac{\left(Ak^{\theta}+(1-\delta)k-\gamma\varkappa k'\right)^{1-\sigma}-1}{1-\sigma}+(\beta\gamma\varkappa)E_{s'|s}v(k',s')\right]$

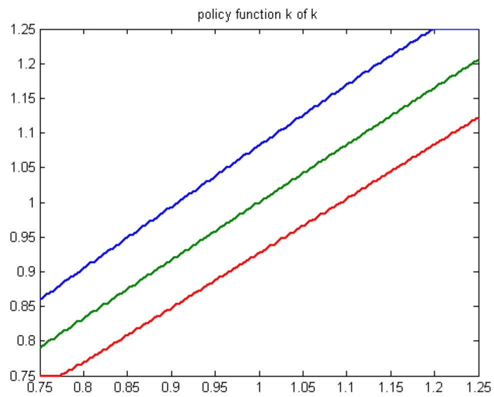
Recursive competitive equilibrium is the value function $v(k,s)$, the policy correspondence $h(k,s)$ and the law of motion of the capital stock $H(K,s)$, such that:

1) $v(k,s)$ satisfies the Bellman equation and $h(k,s)$ is the corresponding optimal policy

2) The law of motion satisfies $H(K,s)=nh\left(\frac{K}{n},s\right)$.

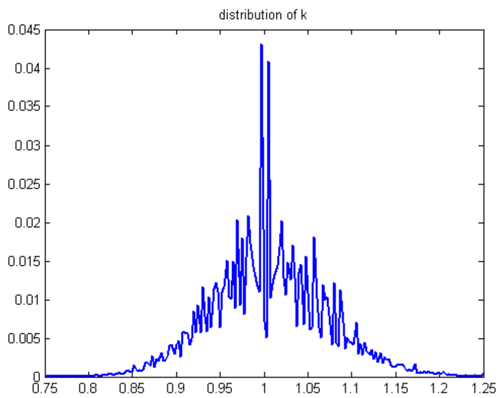
**4-5)** Numerical solution for: $v(k,s)=\max_{k'}\left[\alpha(s)\frac{\left(Ak^{\theta}+(1-\delta)k-\gamma\varkappa k'\right)^{1-\sigma}-1}{1-\sigma}+(\beta\gamma\varkappa)E_{s'|s}v(k',s')\right]$



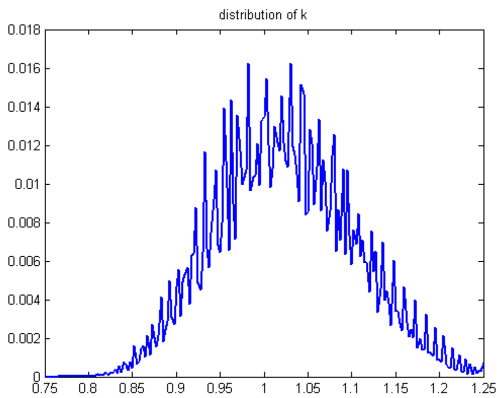




.

To find the unconditional correlation between consumption and investment we need to find the stationary distribution of capital and of states of nature.




(4)



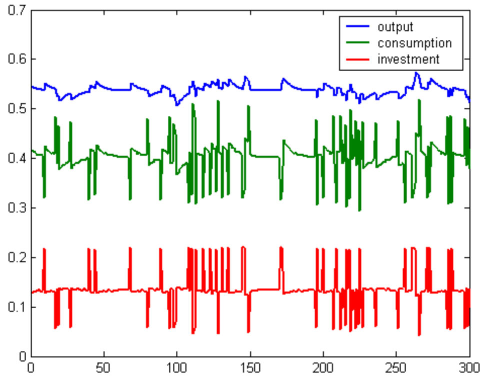

(5)

Then we can easily compute the theoretical correlation between investment and consumption. It is equal -0.96 for both cases. First moments include:

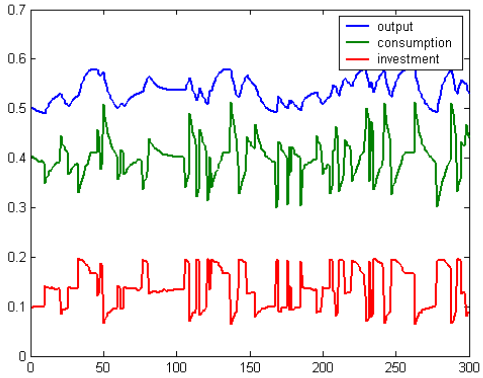
| | Ek | Vk | Ec | Vc | Ei | Vi | Ey | Vy |
|---|---|---|---|---|---|---|---|---|
| (4) | 1.0059 | 0.0043 | 0.4023 | 0.0017 | 0.1349 | 0.0015 | 0.5372 | 0.0001 |
| (5) | 1.0237 | 0.0060 | 0.4030 | 0.0023 | 0.1373 | 0.0021 | 0.5403 | 0.0002 |

Unconditional correlation matrices are essentially the same for any P:

| | i | c | y |
|---|---|---|---|
| i | 1 | -0.96 | -0.08 |
| c | -0.96 | 1 | 0.355 |
| y | -0.08 | 0.355 | 1 |




(4)




(5)

We check the results by Monte-Carlo simulations. The correlation matrices are different both for case (4) and for case (5). The results in simulations could differ a little bit, because to achieve all possible situations we need a very long sample, but in the second case the differences can be attributed to that.

| | i | c | y |
|---|---|---|---|
| i | 1 | -0.96 | -0.08 |
| c | -0.96 | 1 | 0.36 |
| y | -0.08 | 0.36 | 1 |

(4)

| | i | c | y |
|---|---|---|---|
| i | 1 | -0.81 | 0.33 |
| c | -0.81 | 1 | 0.28 |
| y | 0.33 | 0.28 | 1 |

(5)

We get somewhat intuitive results in simulations. If the matrix is closer to diagonal, the variance of capital, and, hence, output is higher. Therefore the components of output are less correlated in absolute value.

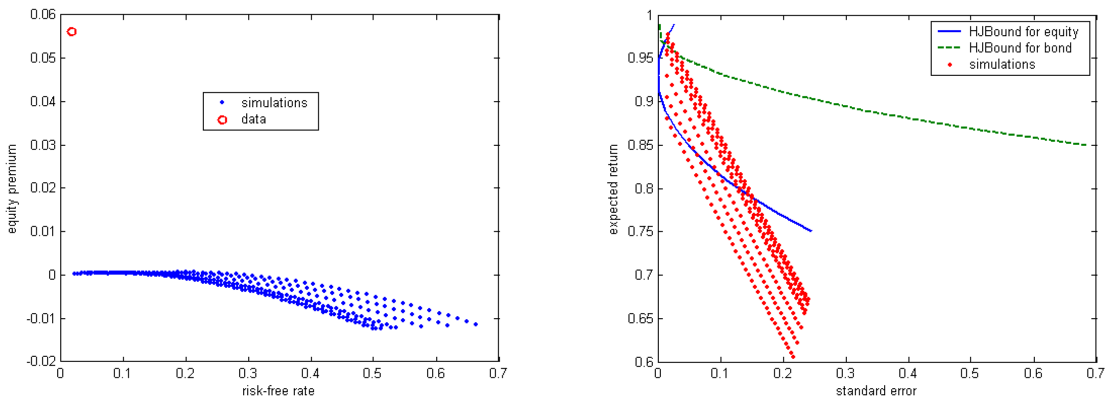
**Exercise 2**

To calculate equity returns we use Standard and Poors Composite Stock Price Index and the Dividends to S&P. As risk-free rates we use Short-Term (1 year) Bonds (Treasury Bills). These three we take from Robert Shiller's addition to chapter 26 of his book. The data on consumption is not updated there. To calculate the last 5 values for consumption growth we use per capita real consumption in 2000 dollars. This is calculated using National Economic Accounts from www.bea.gov.

To derive real equity returns Robert Shiller deflates the S&P index by CPI and adds the growth rate of the index to the present value of real discounted S&P dividends. The real risk-free returns were found by deflating the interest rates on TB's. Using these data we calculate the parameters:

| $\mu = E\Delta c$ | $\delta = \sqrt{V\Delta c}$ | $\phi = (1 + corr(\Delta c, \Delta c'))/2$ | $\lambda_1 = 1 + \mu + \delta$ | $\lambda_2 = 1 + \mu - \delta$ |
|---|---|---|---|---|
| 0.0231 | 0.0170 | 0.5913 | 1.0061 | 1.0401 |

The return to real equity is 0.0749 with st.er. 0.1525, while short-term real risk-free return is 0.0188 with st.er. 0.0262. Following the Mehra-Prescott paper the average equity premium is 5.61%. The important differences from Mehra-Prescott paper for the period 1950-2005 include a reduction in volatility of consumption growth by a factor of 2 and a 1% decrease in the equity premium.




These parameter values could not be matched with the data using any combination of $\beta$ and $\alpha$. However, if we construct Hansen-Jaganattan Bounds, we can see that the data could be satisfactorily matched for unbelievably high $\beta = 0.995$ and $\alpha = 1.5$.