Economics 202 | Matthias Doepke
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Problem Set 2

Labor Supply and Uncertainty

Consider a one-period economy with N agents and M factories. Agents have preferences of the form

$$U(c, n) = u(c) - v(n)$$

over consumption c and labor supply n. The function $u : \mathbf{R}_+ \rightarrow \mathbf{R}$ is bounded, continuously differentiable, strictly increasing, and strictly concave, with $\lim_{c \rightarrow 0} u'(c) = \infty$. The function $v : \mathbf{R}_+ \rightarrow \mathbf{R}$ is bounded, continuously differentiable, strictly increasing, and strictly convex.

Each of the M firms operates a technology described by the production function

$$F_m(\epsilon, n) = \epsilon_m f(n).$$

Here the function $f : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is continuously differentiable, strictly increasing, and strictly concave, with $f(0) = 0$, $\lim_{n \rightarrow 0} f'(n) = \infty$, and $\lim_{n \rightarrow \infty} f'(k) = 0$. ϵ_m is a nonnegative productivity shock. The vector $\epsilon \equiv (\epsilon_1, \epsilon_2, \ldots, \epsilon_M)'$ can take L different values, and the probability of vector ϵ^l is given by $p(l)$. Obviously, we have $p(l) \geq 0$ and $\sum_{l=1}^{L} p(l) = 1$.

Question 1:
Describe this economy in the style of the "Theory of Value." What is the commodity space? What is the consumption set? What is the aggregate production set? What is the resource constraint? (Hint: No lotteries are required here, but you need different commodities for each state of nature.)

Question 2:
Define a competitive equilibrium and show that the conditions for the First and Second Welfare Theorem are satisfied.

Question 3:
Since the Welfare Theorems are satisfied, we can solve for equilibrium allocations by solving a Pareto planning problem for the economy. Write down the Pareto problem and derive the first-order conditions. What can be inferred about the properties of an equilibrium? Among other things, make sure you determine whether the sharing rules for consumption and labor are linear. If your answer is no, which additional assumptions would ensure that the answer is yes? Prove your claims.

Question 4:
Assume you have found an equilibrium allocation. Use this allocation to find an equilibrium price system.

Lotteries and Indivisibilities in General Equilibrium

There is measure one of ex ante identical people. A person can stay at home, work in the market sector, or be a manager. If a person manages n workers, the output of that person's production unit is:

$$n^{\theta},$$

where $0 < \theta < 1$. The utility function of an individual is:

$$\log c - v(h),$$

where $c > 0$ and $h \in \{0, 0.4, 0.7\}$. If a person stays at home, we have $h = 0$. For a worker we have $h = 0.4$, and for a manager we get $h = 0.7$.

Use commodity space R^3 for this competitive analysis. x_1 is consumption; x_2 is the measure or probability of being a worker; and x_3 is the measure or probability of being a manager.

Question 5:
(a) Specify the consumption possibility set X and the agents' utility function which maps X into R.

(b) Specify the aggregate production possibility set Y.

(c) State the First Welfare Theorem and show that its conditions are satisfied for this problem.

(d) Impose a condition on the $v(0)$, $v(0.4)$ and $v(0.7)$ that ensures that $x_2 + x_3 < 1$ (i.e., some people don't work in equilibrium).

An Economy where Experience Matters

Question 6:

(a) Consider an economy with mass one of identical consumers whose preferences are defined by the utility function:

$$\sum_{t=0}^{\infty} \beta^t \left[\log(c_t) - n_t\right],$$

where c_t is consumption, n_t is labor supply, and the parameter β satisfies $0 < \beta < 1$. Labor supply is restricted to lie in the interval (0,1). There is a competitive industry which operates the production technology:

$$y_t = k_t^\alpha N_t^{1-\alpha},$$

where k_t is capital, N_t is aggregate labor supply, and the parameter α satisfies $0 < \alpha < 1$. Capital depreciates completely every period, i.e., the law of motion for capital is $k_{t+1} = i_t$, where i_t is investment. The initial level of capital is given by k_0. In this version of the economy, the market-clearing condition for labor is given by $N_t = n_t$.

- Provide a Bellman equation for the social planning problem.
- Define a sequence-of-markets equilibrium for this economy.
- Find the steady-state levels of capital and labor supply.

(b) So far, we have implicitly assumed that the quality of labor supplied by the households is always the same. In this part of the question, we will assume that the quality of labor depends on prior labor market experience. The experience of a given household is given by e_t. The stock of experience depreciates at rate δ, but it can be increased through work. The law of motion for experience is given by:

$$e_{t+1} = (1 - \delta)e_t + n_t^2.$$

Notice that experience is convex in labor supply, that is, experience increases faster the longer a given person works.

- Provide a Bellman equation for the problem solved by a benevolent social planner under the restriction that all households work the same amount of time (in this case, aggregate labor supply is given by $N_t = e_t n_t$).
- Now consider an alternative arrangement with lotteries such that, ex post, some people work full-time ($n_t = 1$), while others do not work at all ($n_t = 0$). Explain whether it would be possible to improve upon the optimal allocation where everybody is treated identically with such an arrangement. What is the optimal outcome, i.e., will consumption vary across workers and non-workers, and will the same people be working in every period?

Computing Private Information Problems

Private information problems can often be conveniently computed as linear programs. In this problem set you will use linear programming to compute solutions to a standard principal-agent problem. You can work in groups of up to three students and hand in one answer for the whole group.

There is a planner who hires an agent to operate a machine. The agent has reservation utility w_0. The agent exercises effort a, which is unobservable to the planner. Depending on effort, the probability distribution over output q varies. The probability that output q is realized given effort a is $p(q|a)$. We assume that this probability is positive for all combinations of q and a. Depending on the output q, the planner assigns consumption c to the agent, possibly at random. The agent derives utility $U(c, a)$ from consumption c and effort a. The planner is risk neutral and consumes $q - c$. To make computations possible, we assume that there are finite grids A, Q, and C for effort, output, and consumption. We can make the grids very fine to approximate a continuum of choices.

We are now going to formulate the planner's problem as a linear programming problem. The planner offers a contract $\pi(a, q, c)$ to the agent. Here $\pi(a, q, c)$ is the probability that the effort level is a, output is q, and the consumption of the agent is c. The a in this contract can be understood as a recommended effort. The planner cannot force the agent to put in a specific effort, since a is unobservable. The planner chooses a contract that maximizes expected consumption. The planner's problem is:

$$\max_{\pi} \sum_{A,Q,C} \pi(a, q, c)[q - c].$$

We assume that the planner has other sources of income, therefore we do not require the consumption of the planner to be nonnegative. The maximization is subject to a number of constraints. First, the contract has to be a probability distribution. We therefore require $\pi(a, q, c) \geq 0$ for all (a, q, c), and:

$$\sum_{A,Q,C} \pi(a, q, c) = 1. \tag{1}$$

Next, we have to make sure that the contract satisfies the exogenously given probabilities $p(\bar{q}|\bar{a})$ of output $\bar{q}$ given effort $\bar{a}$. For all $\bar{a}$ and $\bar{q}$, we require:

$$\frac{\sum_C \pi(\bar{a}, \bar{q}, c)}{\sum_{Q,C} \pi(\bar{a}, q, c)} = p(\bar{q}|\bar{a}),$$

which can also be written as:

$$\sum_{C} \pi(\bar{a}, \bar{q}, c) = p(\bar{q}|\bar{a}) \sum_{Q,C} \pi(\bar{a}, q, c). \tag{2}$$

Actually, for each $\bar{a}$ one of these constraints is redundant. The agent has to be willing to work for the principal, instead of just receiving reservation utility. Therefore the contract has to deliver at least utility w_0 to the agent:

$$\sum_{A,Q,C} \pi(a,q,c)U(c,a) \geq w_0. \tag{3}$$

Finally, we need incentive compatibility constraints to make sure that the agent actually takes the action that the planner recommends. For all a and $\hat{a}$, we require:

$$\sum_{Q,C} \pi(a,q,c)U(c,a) \geq \sum_{Q,C} \pi(a,q,c)\frac{p(q|\hat{a})}{p(q|a)}U(c,\hat{a}). \tag{4}$$

On the left-hand side is the expected utility the agent gets if he follows the recommended action. On the right-hand side is the utility the agent gets by taking action $\hat{a}$ instead of the recommended action a. This alternative action changes the probability distribution over output.

Notice that the objective function and all constraints (1) to (4) are linear in the probabilities. There is one probability constraint (1), $\#Q \times \#A$ consistency constraints (2), one participation constraint (3), and $\#A \times (\#A - 1)$ incentive-compatibility constraints (4). Clearly, it would take more time than you got for this problem set to solve this problem with pencil and paper. Therefore you will have to use a computer.

It is not hard to solve this problem for an arbitrary utility function and large grids for a, q, and c. However, for the purposes of this problem set we will use a simple example. The utility function is given by:

$$U(c,a) = \sqrt{c}\,(1-a).$$

Effort a can take two values, $a = .2$ or $a = .4$. Output can take two values, $q = 1$ or $q = 2$. Consumption can take twenty values, evenly spaced between .1 and 2 (we assume that the consumer needs at least subsistence consumption .1). The conditional probability distributions over output are given by $p(q = 1|a = .2) = .8$, $p(q = 1|a = .4) = .2$, $p(q = 2|a = .2) = .2$, and $p(q = 2|a = .4) = .8$.

Question 7:
Compute solutions to the planner's problem for 20 different values of w_0, ranging from 0 to 1.1.

Question 8:
Repeat your computations for the full information case. That is, assume that the planner can observe effort. You need to compute the same program without the incentive constraints (4).

Question 9:
Describe your results:

- What are the key differences between the constrained and the full-information solution?
- Conditional on effort and output, is the consumption of the agent randomized? Disregard randomizations that are caused by the finite grid only.
- Is the participation constraint binding for all w_0? Why or why not?

Remarks on Computation:
Please submit the programs you used with your answer. You can use any software you want. Here is an overview of all commands that you need if you use Matlab:

`kron`(A, B)	$A \otimes B$ (Kronecker product). Useful for setting up constraints.
`eye`(n)	n-dimensional identity matrix.
`ones`(n, m)	$n \times m$ matrix of ones.
`zeros`(n, m)	$n \times m$ matrix of zeros.
`linspace`(min, max, n)	Row vector with n evenly spaced elements from min to max.
`fliplr`(A)	Flip vector or matrix A from left to right.

You will also need some loop-structure like `while i<20;...end;` to go through the reservation utilities. Element-by-element operations are done by putting a dot in front of the operator: `.*` is element-by-element multiplication, and so on. Basic vector and matrix assignments work like this: `x=[1 2; 3 4];` produces the result:

$$x = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

You also need a linear-programming routine to solve this problem. While not very fast, the Matlab function `linprog` is sufficient for this application. You call `linprog` like this:

```
x=Linprog(f,A,b,Aeq,beq,lb,ub)
```

The routine solves the problem $\min_x f'x$ subject to $Ax \le b$ and $Aeq\ x = beq$. f, b and beq are vectors, A is the matrix of inequality constraints, and Aeq defines the equality constraints. lb is a vector of lower bounds for x (use zeros), and ub is a vector of upper bounds (use ones). Notice that the routine minimizes the objective function, and the constraints are written as less-than-or-equal-to. You will have to write your program so that it fits this formulation.

Finally, the following is all you need to plot results:

```
plot(xvalues,yvalues,'-'); hold on; axis([xlow xhigh ylow yhigh]);
xlabel('Your label for x-axis'); ylabel('Your label for y-axis');
hold off; print -deps2 filename.eps;
```

You will see a solid line ('-') plotting *xvalues* against *yvalues*. The last command prints the graph to the file `filename.eps` in encapsulated PostScript format.