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Problem Set 1

Dynamic Programming: Theory

Question 1:
Solve Exercises 5.1 a.-c. and 5.5 in Stokey and Lucas.

Question 2:
Consider the following variant of the monk's problem (i.e., a consumer who dislikes consumption). The consumer brings the capital stock k into the period. This capital stock can be divided between consumption c and savings s. Savings own the rate of return $R = \frac{1}{\beta}$, so that:

$$k' = \frac{s}{\beta}.$$

The sequential formulation of the decision problem is:

$$\sup_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} -c_t$$

s.t.

$$c_t \geq 0,$$
$$k' = \frac{k_t - c_t}{\beta}.$$

This problem leads to the Bellman equation:

$$v(k) = \sup_{k' \leq \frac{k}{\beta}} \left\{ -k + \beta k' + \beta v(k') \right\}.$$

This functional equation has a continuum of solutions (show this). Yet, at first sight it appears as if the Bellman equation satisfies the Blackwell conditions, which would imply that the Contraction Mapping Theorem is applicable, in which case there should be a unique solution. Resolve this mystery:

1. Explain exactly where and why the Blackwell conditions and/or the Contraction Mapping Theorem do not apply here. Be specific about the underlying state space, the function space on which the functional equation operates, and so on.

2. Can you specify the problem (again, by choosing appropriate spaces) such that the Blackwell conditions and/or the Contraction Mapping Theorem do apply, leading to a unique (and correct) solution?

Question 3: Consider an economy with a representative consumer whose preferences are described by:

$$\sum_{t=0}^{\infty} \beta^t \frac{c^{1-\sigma}}{1-\sigma}.$$

1. (Two-sector neoclassical growth model)
   Assume that there are separate technologies for producing the consumption good and the investment good. Capital and labor can be moved freely across the two sectors. The resource constraints are given by:

$$\begin{aligned} c_t &\leq l_c^{\alpha} k_c^{1-\alpha} \\ i_t &\leq l_i^{\gamma} k_i^{1-\gamma} \\ l_{it} + l_{ct} &\leq 1 \\ k_{it} + k_{ct} &\leq k_t \\ k_{t+1} &= (1-\delta) k_t + i_t \end{aligned}$$

   Write down the sequential planning problem for this economy. What is the state variable? Write down the Bellman equation for the recursive formulation of the planning problem. Define a state space X that ensures that the utility function is bounded on the state space, and show that the operator defined by the Bellman equation satisfies Blackwell's sufficient conditions for a contraction. Assume that the value function is differentiable, and derive and interpret the first-order and envelope conditions that characterize the optimal solution. Show how you can use the conditions to find the steady state of the economy.

2. (Two sector-model with immobile capital) Now consider the same model under the additional assumption that there are two different types of capital, one used in the consumption goods sector and one in the investment goods sector. The two types of capital cannot be transformed into each other. The resource constraints are:

$$\begin{aligned} c_t &\leq l_c^{\alpha} k_c^{1-\alpha} \\ i_{it} + i_{ct} &\leq l_i^{\gamma} k_i^{1-\gamma} \\ l_{it} + l_{ct} &\leq 1 \\ k_{it+1} &= (1-\delta) k_{it} + i_{it} \\ k_{ct+1} &= (1-\delta) k_{ct} + i_{ct} \end{aligned}$$

   What is the state variable for this economy? Write down the Bellman equation corresponding to the planning problem.

Dynamic Programming: Practice

Question 4: Consider the sequential planning problem:

$$\max \sum_{t=0}^{\infty} (0.6)^t \log c_t.$$

subject to:

$$c_t + k_{t+1} = 15k_t^{0.3} + .5k_t.$$

1. Formulate the Bellman equation for this economy.

2. Use discrete-space dynamic programming to compute the value and policy functions numerically. That is, restrict the state space to the discrete grid (0.05, 0.10, ..., 11.95, 12). Use the initial guess $v_0(k) = 0$ for all k, and iterate on the Bellman equation to compute further guesses of the value function (you need to use a computer for this). Continue until:

$$\max |v_{n+1}(k) - v_n(k)| < 10^{-5}.$$

3. Plot the resulting value and policy functions, and determine the steady-state level of capital.