Econ202B Homework #2 Anton Cheremukhin

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Exercise 1 *Labor supply and uncertainty*

$U(c,n) = u(c) - v(n) \qquad i = \overline{1,N}$
$u : R^+ \to R, \qquad u \in B(x) \cap C^1(x), \qquad u' > 0, u'' < 0, \qquad u'(0) = \infty$
$v : R^+ \to R, \qquad v \in B(x) \cap C^1(x), \qquad v' > 0, v'' > 0$
$F_j(\epsilon_j, n) = \epsilon_j f(n) \qquad j = \overline{1,M} \qquad \epsilon_j^l \geq 0, \qquad p(l) \geq 0, \qquad \Sigma_{l=1}^L p(l) = 1, \qquad l = \overline{1L}$
$f : R^+ \to R^+, \qquad f \in C^1(x), \qquad f' > 0, f'' < 0, \qquad f'(0) = \infty, f'(\infty) = 0$
(1) According to Debreu commodities are determined by time, location and events.
So the commodity space consists of consumption and labor in each state of nature
$S = \{c^1, ..., c^L, n^1, ..., n^L\} = R^{2L}$
Consumption set $X_i = \{x_i \in S | \, x_i^l \geq 0, x_i^{l+L} \leq 0, l = \overline{1L}\}$
Production set $Y_j = \{y_j \in S | \, y_j^l \geq 0, y_j^{L+l} \leq 0, y_j^l \leq \epsilon_j^l f(-y_j^{L+l}), l = \overline{1L}\}$
$\precsim_i$: $U(x_i) = \Sigma_{l=1}^L p(l) \left[u(x_i^l) - v(-x_i^{L+l})\right]$
$\Theta_j : \theta_{ij} \geq 0, \qquad \Sigma_{i=1}^N \theta_{ij} = 1$
Resource constraint $\Sigma_{i=1}^N x_i \leq \Sigma_{j=1}^M y_j$ Economy: $E = \left\{(X_i, \precsim_i)_{i=1}^N, (Y_j, \Theta_j)_{j=1}^M, \right\}$
(2) Competitive equilibrium is a tuple: $(x_i^*, y_j^*, p^*) \in R^{2L(M+N+1)}$ such that
a) $x_i^* = \arg\max_{x_i \in X_i} \{U(x_i) | \, p^* x_i \leq \Sigma_{j=1}^M \theta_{ij} p^* y_j^*\}$
b) $y_j^* = \arg\max_{y_j \in Y_j} p^* y_j$ c) $\Sigma_{i=1}^N x_i^* = \Sigma_{j=1}^M y_i^*$
Conditions for the welfare theorems are:
1stWTh: X_i convex, U - q.concave, no satiation.
2ndWTh: X_i convex, finite demetsional, U - continuous, q.concave, no satiation, Y_j convex.

By definition of X_i it is convex. Taking into account that $f(.)$ is strictly increasing and strictly concave and by definition of Y_j it is convex. Functions $u(.)$ and $v(.)$ being continuously differentiable implies continuity, and being strictly increasing - no satiation of $U(.)$. Moreover a sum of strictly concave functions $u(.)$ and $-v(.)$ is strictly quasi-concave which accomplishes the proof.

Since X_i is bounded below, closed and connected, preferences are continuous, Y_j is closed and convex, and gives no free lunch the optimum exists. Since X_i is non-empty, bounded below, closed and convex, preferences are continuous, quasi-concave and exhibit no satiation, Y_j is closed and convex, production allows inaction, is irreversible and allows free disposal the equilibrium also exists. Therefore, we can first characterize the optimum which will coincide with equilibrium allocations.

(3) $\max_{x_i \in X_i, y_j \in Y_j} \left\{\Sigma_{i=1}^N \phi_i \Sigma_{l=1}^L p(l) \left[u(x_i^l) - v(-x_i^{L+l})\right] \middle| \quad \Sigma_{i=1}^N x_i \leq \Sigma_{j=1}^M y_j, \quad y_j^l \leq \epsilon_j^l f(-y_j^{L+l})\right\}$

Let's substitute (for the sake of simplicity) c_i^l and n_i^l instead of x_i^l and $-x_i^{l+L}$ for consumers and n_j^l instead of $-y_j^{l+L}$ for producers. Then the Lagrangian is:
$\max_{c_i^l, n_i^l, n_j^l} \Sigma_{l=1}^L \left(p(l) \Sigma_{i=1}^N \phi_i \left[u(c_i^l) - v(n_i^l)\right] + \lambda^l \left[\Sigma_{i=1}^N n_i^l - \Sigma_{j=1}^M n_j^l\right] + \mu^l \left[\Sigma_{j=1}^M \epsilon_j^l f(n_j^l) - \Sigma_{i=1}^N c_i^l\right]\right)$
This leads to the following first order conditions:
$\phi_i p(l) u'(c_i^l) = \mu^l \qquad \phi_i p(l) v'(n_i^l) = \lambda^l \qquad \lambda^l = \mu^l \epsilon_j^l f'(n_j^l) \Rightarrow \qquad \frac{\lambda^l}{\mu^l} = \frac{v'(n_i^l)}{u'(c_i^l)} = \epsilon_j^l f'(n_j^l)$
$\phi_i p(l) u'(c_i^l) = \mu^l = \phi_{-i} p(l) u'(c_{-i}^l) \qquad \phi_i p(l) v'(n_i^l) = \lambda^l = \phi_{-i} p(l) v'(n_{-i}^l)$

The assumption of CES utility functions for both consumption and labor ensure that the consumption and labor sharing rules are linear:

$\phi_i \left(c_i^l\right)^{-\sigma} = \phi_{-i} \left(c_{-i}^l\right)^{l-\sigma}, \phi_i \left(n_i^l\right)^{-\rho} = \phi_{-i} \left(n_{-i}^l\right)^{-\rho} \qquad \Rightarrow \qquad c_i^l = \left(\frac{\phi_i}{\phi_{-i}}\right)^{1/\sigma} c_{-i}^l, n_i^l = \left(\frac{\phi_i}{\phi_{-i}}\right)^{1/\rho} n_{-i}^l$

Another possible assumption is that the central planner assigns equal weights to all consumers ($\phi_i = \phi_{-i}$), than all consumers will be offered identical behavior independent of the form of their utility function: $n_i^l = n_{-i}^l \qquad c_i^l = c_{-i}^l$.In this case labor and consumption will be equally shared between consumers in each state. However the formula above for the sharing rules is true in both cases. We can use it to find the solution:

$n_{i_0}^l \phi_{i_0}^{-1/\sigma} \Sigma_{i=1}^N \phi_i^{1/\sigma} = \Sigma_{j=1}^M n_j^l \qquad c_{i_0}^l \phi_{i_0}^{-1/\rho} \Sigma_{i=1}^N \phi_i^{1/\rho} = \frac{1}{N} \Sigma_{j=1}^M \epsilon_j^l f\left(n_j^l\right)$

$\frac{v'\left(n_{i_0}^l\right)}{u'\left(c_{i_0}^l\right)} = \frac{v'\left(\left[\Sigma_{j=1}^M n_j^l\right] / \left[\phi_{i_0}^{-1/\rho} \Sigma_{i=1}^N \phi_i^{1/\rho}\right]\right)}{u'\left(\left[\Sigma_{j=1}^M \epsilon_j^l f\left(n_j^l\right)\right] / \left[\phi_{i_0}^{-1/\sigma} \Sigma_{i=1}^N \phi_i^{1/\sigma}\right]\right)} = \epsilon_j^l f'\left(n_j^l\right), \qquad j = \overline{1, M}, l = \overline{1, L}, i_0 = \overline{1, N}$

So we have (2N+M)L equations and (2N+M)L unknowns. The solution is unique sinse l.h.s. of the last equation is strictly increasing in each argument n_j^l while the r.h.s. is strictly decreasing ($v'' > 0, u'' < 0, f' > 0, f'' < 0$). Knowing this unique solution we could solve backwards to find equilibrium consumption and labor. This gives us $L\left(2N+M\right)$ equilibrium values.

In general we get $L(M+2N+2)$ equations with $L\left(M+2N+2\right)$ unknowns which due to Inada conditions have a unique solution: $\qquad \{n_j^l, n_i^l, c_i^l, \lambda^l, \mu^l\} \qquad \Sigma_{i=1}^N n_i^l = \Sigma_{j=1}^M n_j^l$

$\phi_i p\left(l\right) u'\left(c_i^l\right) = \mu^l \qquad \phi_i p\left(l\right) v'\left(n_i^l\right) = \lambda^l \qquad \lambda^l = \mu^l \epsilon_j^l f'\left(n_j^l\right) \qquad \Sigma_{j=1}^M \epsilon_j^l f\left(n_j^l\right) = \Sigma_{i=1}^N c_i^l$

(4)

$L\left(x_i\right) = \Sigma_{l=1}^L p\left(l\right)\left[u\left(c_i^l\right) - v\left(n_i^l\right)\right] + \lambda_i \Sigma_{l=1}^L \left[\Sigma_{j=1}^M \theta_{ij}\left(p_c^l y_j^l - p_n^l n_j^l\right) + p_n^l n_i - p_c^l c_i\right]$

FOC: $\qquad p\left(l\right) u'\left(c_i^l\right) = p_c^l \lambda_i \qquad p\left(l\right) v'\left(n_i^l\right) = p_n^l \lambda_i \qquad (N * L * 2 \text{ equations})$

Restating gives: $\frac{p(l)u'\left(c_i^l\right)}{p_c^l} = \lambda_i = \frac{p(l)v'\left(n_i^l\right)}{p_n^l}$ which implies:

$p_c^l = p_c^1 \frac{p(l)u'\left(c_i^l\right)}{p(1)u'\left(c_i^1\right)} \qquad p_n^1 = p_c^1 \frac{v'\left(n_i^1\right)}{u'\left(c_i^1\right)} \qquad p_n^l = p_n^1 \frac{p(l)u'\left(n_i^l\right)}{p(1)u'\left(n_i^1\right)} = p_c^1 \frac{v'\left(n_i^1\right)}{u'\left(c_i^1\right)} \frac{p(l)u'\left(n_i^l\right)}{p(1)u'\left(n_i^1\right)}$

Putting the price of consumption in state 1 $p_c^1 = 1$ assigns a numerair, which determines the rest of the prices.

In general in the competitive setting an analog of central planner's weights are the shares in firms outputs. The budget constraints give us N equations:

$\Sigma_{l=1}^L \Sigma_{j=1}^M \theta_{ij}\left(p_c^l y_j^l - p_n^l n_j^l\right) = \Sigma_{l=1}^L \left[p_c^l c_i - p_n^l n_i\right]$

From the firm's problem it follows that $\qquad \epsilon_j^l f'\left(n_j^l\right) = \frac{p_n^l}{p_c^l} \qquad (M * L \text{ constraints})$

Resource constraints imply: $\qquad \Sigma_{j=1}^M \epsilon_j^l f\left(n_j^l\right) = \Sigma_{i=1}^N c_i^l \qquad \Sigma_{i=1}^N n_i^l = \Sigma_{j=1}^M n_j^l \ (2 * L \text{ constraints})$

The whole system has $L\left(M+2N+2\right)$ equations with the same number of unknowns, and due to Inada conditions has a unique solution. The only free parameter left is the choice of numeraire price. It is easy to show that $\lambda^l = p_n^l$ and $\mu^l = p_c^l$ is one of the price solutions. In this case the numeraire is specified by the fact that probabilities of states sum up to one. In the competitive setting the sharing rules depend on the distribution of wealth. An analogous situation of linear sharing rules arizes if all agents have equal wealth, i.e. equal shares in the firm's profits:

$\theta_{ij} = \frac{1}{N} \qquad \Rightarrow \qquad n_i^l = n_{-i}^l \equiv n^l \qquad c_i^l = c_{-i}^l \equiv c^l$

Exercise 2 *Lotteries and indivisibilities in general equilibrium*

(a) Commodity space $S = \{x_1, x_2, x_3\} = R^3$

Consumption set $X = \{x \in S|\, x_1 \geq 0, x_2 \leq 0, x_3 \leq 0, -x_2 - x_3 \leq 1\}$

$\precsim_i$: $U(x) = \log(x_1) - (1 + x_2 + x_3)\, v(0) + x_2 v(0.4) + x_3 v(0.7) =$

$\log(x_1) + x_2 (v(0.4) - v(0)) + x_3 (v(0.7) - v(0)) - v(0) = \log(x_1) - a(-x_2) - b(-x_3) - c$

(b) Fraction $-x_2$ are workers, fraction $-x_3$ are managers. Hence, each of the $-x_3$ managers produces $n^\theta = (x_2/x_3)^\theta$ output. The total output is $-x_3 (x_2/x_3)^\theta = (-x_2)^\theta (-x_3)^{1-\theta}$.

Production set $Y = \left\{ y \in S|\, y_2 \leq 0, y_3 \leq 0, 0 \leq y_1 \leq (-y_2)^\theta (-y_3)^{1-\theta} \right\}$

Resource constraint $X \leq Y$ Economy: $E = \{(X, \precsim), (Y)\}$

(c) Competitive equilibrium is a tuple: $(x^*, y^*, p^*) \in R^{3*3}$ such that

a) $x^* = \arg\max_{x \in X} \{U(x)|\, p^* x \leq p^* y^*\}$

b) $y^* = \arg\max_{y \in Y} p^* y$ c) $x^* = y^*$

Conditions for the first welfare theorem are: X convex, U - q.concave, no satiation.

By definition of X it is convex. Functions $\log(x)$, ax and bx are strictly increasing - no satiation of $U(.)$. A sum of a strictly concave function $\log(.)$ and a linear function is quasi-concave which accomplishes the proof.

(d) $\max_{x_i} \left\{ \log(x_1) - a x_2 - b x_3 - c \right| \quad x_1 \leq x_2^\theta x_3^{1-\theta}, \quad x_2 + x_3 \leq 1, \quad x_i \geq 0 \}$

$L = \log(x_1) - a x_2 - b x_3 - c + \lambda \left[x_2^\theta x_3^{1-\theta} - x_1 \right]$

FOC: $\lambda x_1 = 1 \quad a = \lambda x_2^\theta x_3^{1-\theta} \frac{\theta}{x_2} \quad b = \lambda x_2^\theta x_3^{1-\theta} \frac{1-\theta}{x_3}$ RC: $x_1 = x_2^\theta x_3^{1-\theta}$

$x_2 = \frac{\theta}{a}, \quad x_3 = \frac{1-\theta}{b} \quad \Rightarrow x_2 + x_3 = \frac{\theta}{a} + \frac{1-\theta}{b} < 1 \Leftrightarrow \quad \boxed{\frac{\theta}{v(0.4) - v(0)} + \frac{1-\theta}{v(0.7) - v(0)} < 1}$

Exercise 3 *Economy where experience matters*

(a) $\max \left\{ \Sigma_{t=0}^\infty \beta^t [\log c_t - n_t] \right|\, y_t = k_t^\alpha N_t^{1-\alpha}, N_t = \int_0^1 e_t^i n_t^i di = n_t, e_t^i = 1, k_{t+1} = i_t = y_t - c_t \Big\}$

$$\boxed{v(k) = \max_{k', 0 \leq n \leq 1} \{\log[k^\alpha n^{1-\alpha} - k'] - n + \beta v(k')\}}$$

A recursive($\equiv$sequential) competitive($\equiv$market) equilibrium is a set: $\{v(k), k'(k), n(k)\}$- value function and decision rules($\equiv$policy functions), that solve the Bellman equation.

To find the steady state values we need to write the first order conditions and the envelope theorem: $\frac{y}{y-k'} \frac{1-\alpha}{n} = 1 \quad \frac{1}{y-k'} = \beta v'(k') \quad v'(k) = \frac{y}{y-k'} \frac{\alpha}{k}$ and impose $k = k'$.

$1 = \frac{y^*}{k^*} \beta\alpha \quad n^* = \frac{1-\alpha}{1 - \frac{k^*}{y^*}} = \frac{1-\alpha}{1-\alpha\beta} \quad k^* = (\beta\alpha)^{\frac{1}{1-\alpha}} n^* = \frac{1-\alpha}{1-\alpha\beta} (\beta\alpha)^{\frac{1}{1-\alpha}}$ (0)

(b) The case where people are restricted to same behavior:

$\max \left\{ \Sigma_{t=0}^\infty \beta^t [\log c_t - n_t] \right|\, k_{t+1} = k_t^\alpha N_t^{1-\alpha} - c_t, N_t = e_t n_t, e_{t+1} = (1-\delta) e_t + n_t^2 \}$

$n = \sqrt{e' - (1-\delta) e} = n(e, e') \quad c = k^\alpha (e n(e, e'))^{1-\alpha} - k' = c(k, k', e, e')$

$$\boxed{v(k, e) = \max_{k', e'} \left\{ \log\left[k^\alpha (e n(e, e'))^{1-\alpha} - k' \right] - n(e, e') + \beta v(k', e') \right\}}$$

FOC, ENV: $\frac{y}{y-k'} \frac{1-\alpha}{n} n_2'(e, e') - n_2'(e, e') + \beta v_2'(k', e') = 0 \quad \frac{1}{y-k'} = \beta v_1'(k', e')$

$v_2'(k, e) = \frac{y}{y-k'} \left[\frac{1-\alpha}{n} n_1'(e, e') + \frac{1-\alpha}{e} \right] - n_1'(e, e') \quad v_1'(k, e) = \frac{y}{y-k'} \frac{\alpha}{k}$

s.s.: $1 = \alpha\beta \frac{y}{k} \quad \left(\frac{y}{y-k} \frac{1-\alpha}{n} - 1 \right) n_2'(e, e) + \beta \left[\left(\frac{y}{y-k} \frac{1-\alpha}{n} - 1 \right) n_1'(e, e) + \frac{y}{y-k} \frac{1-\alpha}{e} \right] = 0$

$n_1'(e, e) = \frac{-(1-\delta)}{2\sqrt{\delta e}} \quad n_2'(e, e) = \frac{1}{2\sqrt{\delta e}} \quad n(e, e') = \sqrt{\delta e} \quad \frac{y}{y-k} = \frac{1}{1-\alpha\beta}$

$\left(\frac{1-\alpha}{1-\alpha\beta} \frac{1}{\sqrt{\delta e}} - 1 \right) \frac{1}{2\sqrt{\delta e}} [1 - \beta(1-\delta)] + \beta \frac{1-\alpha}{1-\alpha\beta} \frac{1}{e} = 0$

$$n^* = \sqrt{\delta e^*} = \frac{1-\beta+3\beta\delta}{1-\beta+\beta\delta}\frac{1-\alpha}{1-\alpha\beta} > 1 \qquad k^* = (\alpha\beta)^{\frac{1}{1-\alpha}} e^* n^* = \frac{1}{\delta}\left[\frac{1-\beta+3\beta\delta}{1-\beta+\beta\delta}\frac{1-\alpha}{1-\alpha\beta}\right]^3 (\alpha\beta)^{\frac{1}{1-\alpha}} \qquad (1')$$

Hence, $n^* = \sqrt{\delta e^*} = 1 \qquad k^* = (\alpha\beta)^{\frac{1}{1-\alpha}} e^* n^* = \frac{1}{\delta}(\alpha\beta)^{\frac{1}{1-\alpha}}$ (1)

In the setup with lotteries there could be two different settings. In the first setting the central planner flips the coin only once, which determines the workers which work all the time. These working guys are exactly identical and work $n_t = 1$. Their experience changes according to: $e_{t+1} = (1-\delta)e_t + 1$ and has a steady-state level: $e = 1/\delta$. Let the fraction of workers be n.

$\max\left\{\Sigma_{t=0}^{\infty}\beta^t[\log c_t - n] \mid k_{t+1} = k_t^{\alpha}N_t^{1-\alpha} - c_t, N_t = e_t n = n/\delta\right\}$

$$\boxed{v(k) = \max_{k'}\left\{\log\left[k^{\alpha}(n/\delta)^{1-\alpha} - k'\right] - n + \beta v(k')\right\}}$$

The solution to this is: $\frac{k}{y} = \alpha\beta \qquad k^{\alpha}(n/\delta)^{1-\alpha} = y \qquad k^* = \frac{n}{\delta}(\alpha\beta)^{\frac{1}{1-\alpha}}$

The planner aims at maximizing lifetime utility, which is proportional to $\log k^*(n) - n$.

Hence, in optimum $n^* = 1$ and $k^* = \frac{1}{\delta}(\alpha\beta)^{\frac{1}{1-\alpha}}$. (2)

In the other setup the central planner flips a coin and a fraction n of consumers chosen i.i.d. from the sample work 1, while the rest work 0. Since it's a lottery, it must be random and include all agents, which implies that every period they are all ex ante identical everybody has equal chances to work and consumes the same amount.

$\max\left\{E_0\Sigma_{t=0}^{\infty}\beta^t[\log c_t - n_t] \mid k_{t+1} = k_t^{\alpha}N_t^{1-\alpha} - c_t, N_t = \int_0^1 e_t^i n_t^i di, e_{t+1}^i = (1-\delta)e_t^i + (n_t^i)^2\right\}$

Let's aggregate them: $N_t = E\int_0^1 e_t^i n_t^i di = \int_0^1 e_t^i(n*1 + (1-n)*0)\,di = n_t\int_0^1 e_t^i di$

$$N_{t+1} = E\int_0^1 e_{t+1}^i n_{t+1}^i di = E\int_0^1\left[(1-\delta)e_t^i + (n_t^i)^2\right]n_{t+1}^i di =$$

$$= \int_0^1\left[\begin{array}{c} n_{t+1} * 1 * \left[(1-\delta)e_t^i n_{t+1}^i + (n_t^i)^2 n_{t+1}^i\right]_{n_{t+1}^i=1} + \\ (1-n) * \left[(1-\delta)e_t^i n_{t+1}^i + (n_t^i)^2 n_{t+1}^i\right]_{n_{t+1}^i=0}\end{array}\right]di =$$

$$= n_{t+1}\int_0^1\left[(1-\delta)e_t^i + n_t * 1 + (1-n_t)*0\right]di =$$

$$= \frac{n_{t+1}}{n_t}(1-\delta)n_t\int_0^1 e_t^i di + n_t n_{t+1}\int_0^1 1 di = (1-\delta)\frac{n_{t+1}}{n_t}N_t + n_t n_{t+1}.$$

If we think that $n_t = n_{t+1}$ is predetermined, than the number of workers is exogenous to the model. It's dynamics depends on the initial conditions N_0 and is convergent: $N^* = n^2/\delta$. Since we are interested in the steady state, we could just assume that labor supply has already converged to its steady state value. This can be justified by the fact that the planner could choose the initial distribution of working people not exactly randomly, but in such a way that the aggregate amount of labor corresponds to the steady-state level, while flipping the coin in all future periods.

$E_0\Sigma_{t=0}^{\infty}\beta^t[\log c_t^i - n_t^i] = E_0\Sigma_{t=0}^{\infty}\beta^t[\log c_t - n_t*1 - (1-n_t)*0] = \Sigma_{t=0}^{\infty}\beta^t[\log c_t - n]$

It this case the

$$\boxed{v(k) = \max_{k'}\left\{\log\left[k^{\alpha}\left(\frac{n^2}{\delta}\right)^{1-\alpha} - k'\right] - n + \beta v(k')\right\}}$$

FOC: $\frac{1}{y-k'} = \beta v'(k') \qquad v'(k) = \frac{y}{y-k'}\frac{\alpha}{k}$ and impose $k = k'$.

$1 = \frac{y^*}{k^*}\beta\alpha \qquad k^{*\alpha}\left(\frac{n^2}{\delta}\right)^{1-\alpha} = y^* \qquad k^* = \frac{n^2}{\delta}(\beta\alpha)^{\frac{1}{1-\alpha}}$

If we again maximize $\log k^*(n) - n$ from FOC we get $n^* = 2$ which is not allowed. (2′)

Hence, again we have $n^* = 1$ and $k^* = \frac{1}{\delta}(\alpha\beta)^{\frac{1}{1-\alpha}}$. (2)

These two results don't seem reasonable, since it is not really a lottery.

There must be no reason for changing the probability between periods in the steady state, but in general it is possible to change it from period to period.

$N_t = (1-\delta)\frac{n_t}{n_{t-1}}N_{t-1} + n_t n_{t-1} = (1-\delta)^j\frac{n_t}{n_{t-j}}N_{t-j} + n_t\Sigma_{s=1}^{j}(1-\delta)^{s-1}n_{t-s}$

So, aggregating over total labor supply is not a good idea. Denote $\int_0^1 e_t^i di = e_t$.
Then $e_{t+1} = \frac{N_{t+1}}{n_{t+1}} = (1-\delta)\frac{N_t}{n_t} + n_t = (1-\delta)e_t + n_t$ Hence experience is a state variable.
Therefore, the recursive problem in this case is:
$\max\left\{\Sigma_{t=0}^{\infty}\beta^t\left[\log c_t - n_t\right]\middle|\, k_{t+1} = k_t^{\alpha}N_t^{1-\alpha} - c_t, N_t = e_t n_t, e_{t+1} = (1-\delta)e_t + n_t\right\}$
$n = e' - (1-\delta)e = n(e,e')$ $c = k^{\alpha}(en(e,e'))^{1-\alpha} - k' = c(k,k',e,e')$

$$\boxed{v(k,e) = \max_{k',e'}\left\{\log\left[k^{\alpha}(en(e,e'))^{1-\alpha} - k'\right] - n(e,e') + \beta v(k',e')\right\}}$$

FOC, ENV: $\frac{y}{y-k'}\frac{1-\alpha}{n}n_2'(e,e') - n_2'(e,e') + \beta v_2'(k',e') = 0$ $\frac{1}{y-k'} = \beta v_1'(k',e')$
$v_2'(k,e) = \frac{y}{y-k'}\left[\frac{1-\alpha}{n}n_1'(e,e') + \frac{1-\alpha}{e}\right] - n_1'(e,e')$ $v_1'(k,e) = \frac{y}{y-k'}\frac{\alpha}{k}$
s.s.: $1 = \alpha\beta\frac{y}{k}$ $\left(\frac{y}{y-k}\frac{1-\alpha}{n} - 1\right)n_2'(e,e) + \beta\left[\left(\frac{y}{y-k}\frac{1-\alpha}{n} - 1\right)n_1'(e,e) + \frac{y}{y-k}\frac{1-\alpha}{e}\right] = 0$
$n_1'(e,e) = -(1-\delta)$ $n_2'(e,e) = 1$ $n(e,e) = \delta e$ $\frac{y}{y-k} = \frac{1}{1-\alpha\beta}$
$\left(\frac{1-\alpha}{1-\alpha\beta}\frac{1}{\delta e} - 1\right) + \frac{\beta\delta}{1-\beta+\beta\delta}\frac{1-\alpha}{1-\alpha\beta}\frac{1}{\delta e} = 0$ $n^* = \delta e^* = \frac{1-\beta+2\beta\delta}{1-\beta+\beta\delta}\frac{1-\alpha}{1-\alpha\beta} > 1$

$$k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}e^*n^* = \frac{1}{\delta}\left[\frac{1-\beta+2\beta\delta}{1-\beta+\beta\delta}\frac{1-\alpha}{1-\alpha\beta}\right]^2(\alpha\beta)^{\frac{1}{1-\alpha}} \quad (3')$$

Again we get the same outcome: $n^* = 1$ $k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}e^*n^* = \frac{1}{\delta}(\alpha\beta)^{\frac{1}{1-\alpha}}$ (3)

All the outcomes are equivalent, but that's totally due to the restriction on n_t. If we changed everything so that the scale increased by, say, 5, than the initial solutions shoud be compared.
$v(k^*) = \frac{1}{1-\beta}\left[\log\frac{1-\beta\alpha}{\beta\alpha} + \log k^* - n^*\right]$ Hence, need to compare $\log k^* - n^*$ in three states:
$\Delta_1 = v(lottery1) - v(no-lottery) = v(2) - v(1)$
$\log\left(\frac{1}{\delta}(\beta\alpha)^{\frac{1}{1-\alpha}}\right) - \log\left(\frac{1}{\delta}\left[\frac{1-\beta+3\beta\delta}{1-\beta+\beta\delta}\frac{1-\alpha}{1-\alpha\beta}\right]^3(\alpha\beta)^{\frac{1}{1-\alpha}}\right) - 1 + \frac{1-\beta+3\beta\delta}{1-\beta+\beta\delta}\frac{1-\alpha}{1-\alpha\beta} =$
$= \left(1 + \frac{2\beta\delta}{1-\beta+\beta\delta}\right)\frac{1-\alpha}{1-\alpha\beta} - 1 - 3\log\left(\left(1 + \frac{2\beta\delta}{1-\beta+\beta\delta}\right)\left(\frac{1-\alpha}{1-\alpha\beta}\right)\right) = (1+2x)y - 1 - 3\log(1+2x) - 3\log y$
at $\beta \approx 1$ $x = \frac{\beta\delta}{1-\beta+\beta\delta} \approx 1$ $y = \frac{1-\alpha}{1-\alpha\beta} \approx 1$
$\Delta_1 \approx 2 - 3\ln 3 \approx -1.3 < 0$ The no-lottery case is better.
$\Delta_2 = v(lottery2) - v(no-lottery) = v(2) - v(1)$
$\log\left(\frac{4}{\delta}(\beta\alpha)^{\frac{1}{1-\alpha}}\right) - \log\left(\frac{1}{\delta}\left[\frac{1-\beta+3\beta\delta}{1-\beta+\beta\delta}\frac{1-\alpha}{1-\alpha\beta}\right]^3(\alpha\beta)^{\frac{1}{1-\alpha}}\right) - 2 + \frac{1-\beta+3\beta\delta}{1-\beta+\beta\delta}\frac{1-\alpha}{1-\alpha\beta} =$
$= \left(1 + \frac{2\beta\delta}{1-\beta+\beta\delta}\right)\frac{1-\alpha}{1-\alpha\beta} - 2 + \log 4 - 3\log\left(\left(1 + \frac{2\beta\delta}{1-\beta+\beta\delta}\right)\left(\frac{1-\alpha}{1-\alpha\beta}\right)\right) =$
$= (1+2x)y - 3\log(1+2x) - 3\log y - 2 + \log 4$
$\Delta_2 \approx 1 + 2\ln 2 - 3\ln 3 \approx -0.91 < 0$ The no-lottery case is better.
$\Delta_3 = v(lottery3) - v(no-lottery) = v(3) - v(1) =$
$\log\left(\frac{1}{\delta}\left[\frac{1-\beta+2\beta\delta}{1-\beta+\beta\delta}\frac{1-\alpha}{1-\alpha\beta}\right]^2(\alpha\beta)^{\frac{1}{1-\alpha}}\right) - \frac{1-\beta+2\beta\delta}{1-\beta+\beta\delta}\frac{1-\alpha}{1-\alpha\beta} - \log\left(\frac{1}{\delta}\left[\frac{1-\beta+3\beta\delta}{1-\beta+\beta\delta}\frac{1-\alpha}{1-\alpha\beta}\right]^3(\alpha\beta)^{\frac{1}{1-\alpha}}\right) + \frac{1-\beta+3\beta\delta}{1-\beta+\beta\delta}\frac{1-\alpha}{1-\alpha\beta}$
$= \frac{\beta\delta}{1-\beta+\beta\delta}\frac{1-\alpha}{1-\alpha\beta} - \log\left(\left(\frac{1-\beta+3\beta\delta}{1-\beta+\beta\delta}\right)\left(\frac{1-\alpha}{1-\alpha\beta}\right)\right) = xy - 3\log(2x+1) + 2\log(x+1) - \log y$ vs 0
$\Delta_3 \approx 1 - 3\ln 3 + 2\ln 2 \approx -0.91 < 0$ The no-lottery case is better.

In all cases the gains from an increase in expected steady state value of experience due to convexifying experience growth is outweighed by the loss of labor supply due to an increase in variance of experience, induced by the lottery.

Exercise 4 *Private Information*

(by Anton Cheremukhin, Paulina Restrepo Echavarria, Hisayuki Yoshimoto)

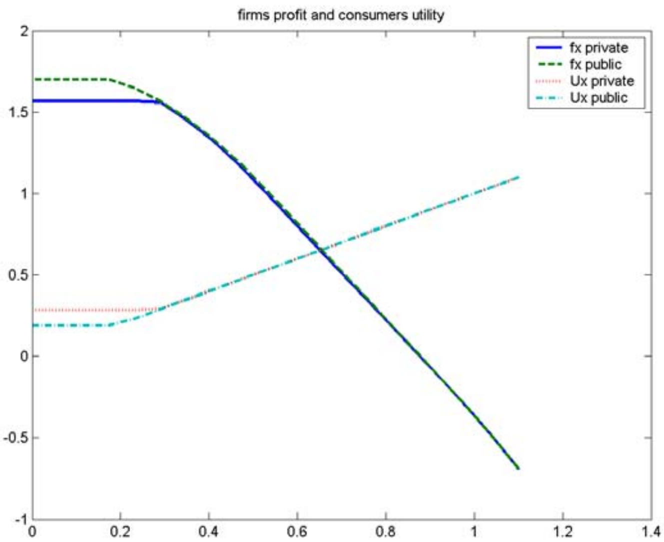
The first 40 graphs (one for each outside utility level and for the public and private settings) plot probabilities that the agent gets consumption c for 4 outcomes. The next 8 graphs aggregate the probabilities and expected utilities of outcomes (effort levels and output levels), taking expectations over the consumption grid.

The resulting policy could be interpreted if we say different words about the structure. The policy is a contract proposed to the agent in period 1. He is also suggested the level of efforts to take. In period 2 given the contract the agent decides whether he should take a different effort all other things equal. In period 3 the output is created. In period 4 the firm flips a coin according to the policy and under the assumption that the agent told the truth and gives the agent some level of consumption. The incentive compartibility constraint is exactly the condition that the agent won't lie given the policy, other agents' actions and existence of commitment to it.

When outside opportunities are low the firm wants all workers to work hard. In the case of full information it does not pay anything for high level of effort. For the private information case the firm has to pay workers some minimum amount to encourage them not to lie. Workers receive their information rent. This divergence of payments for different output among high-effort workers remains for all levels of outside opportunities just to encourage them to tell the truth. There is no divergence in payments among low-effort workers.

As outside opportunities increase the payments rize. At the level of outside utility of about 0.6 some workers are given an advice to be lazy. At the level of 0.8 most workers are lazy. There is no place for high effort at the level of 1. This happens because the firm has to encourage the workers to participate at all, since this gives the firm at least output of 1. In this case the firm just cannot afford high levels of efforts. The dinamics of efforts and output are reflected in the graphs.

The very first graph shows the differences in utilities and profits between the public and private information cases. It demonstrates the existence of information rent for low outside opportunities. For all cases randomization is low: agents are basicly given a discrete choice. This is a good illustration of the revelation principle, as simple non-stochastic actions are prescribed to all four combinations of types and outcomes.



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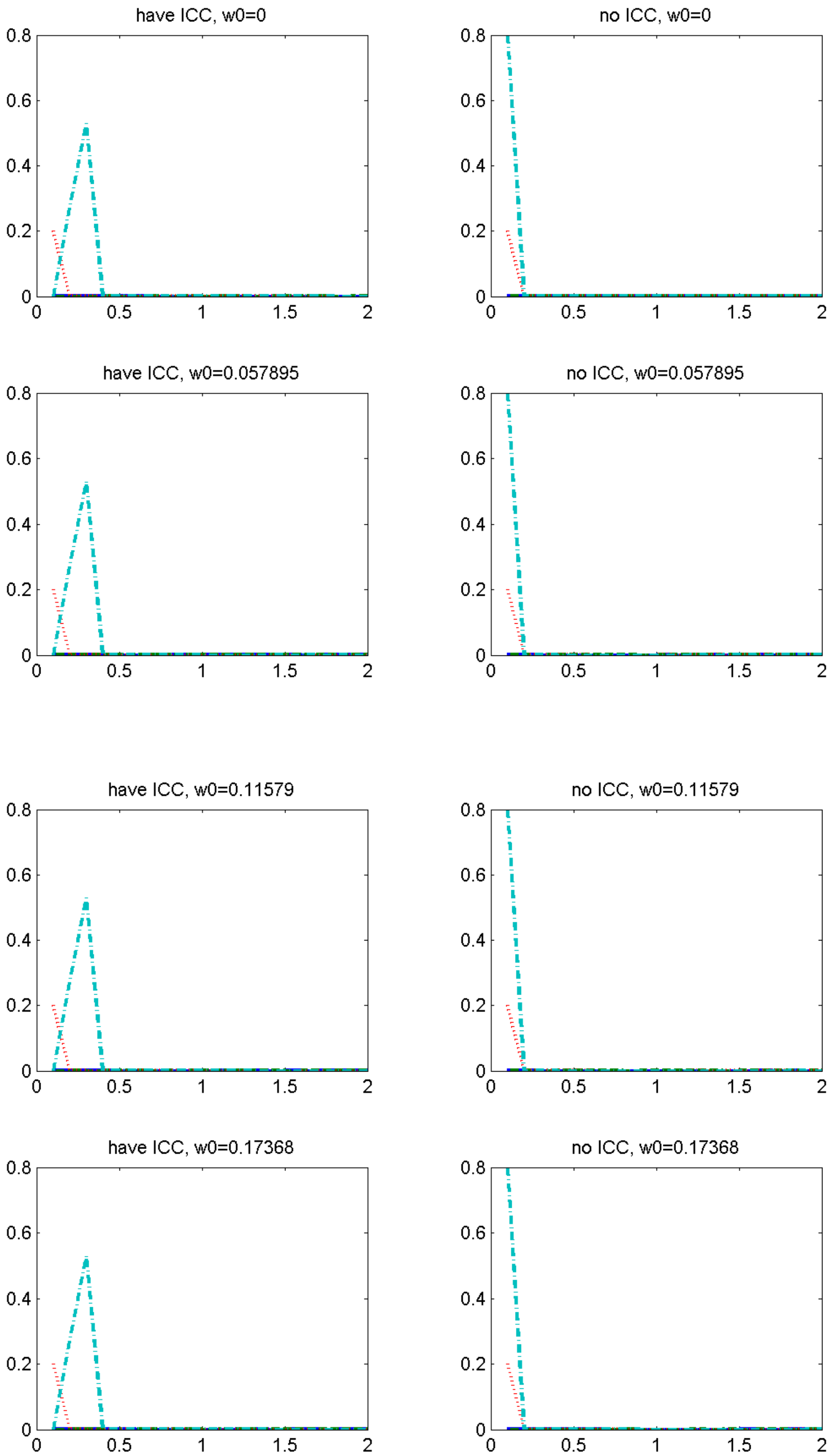
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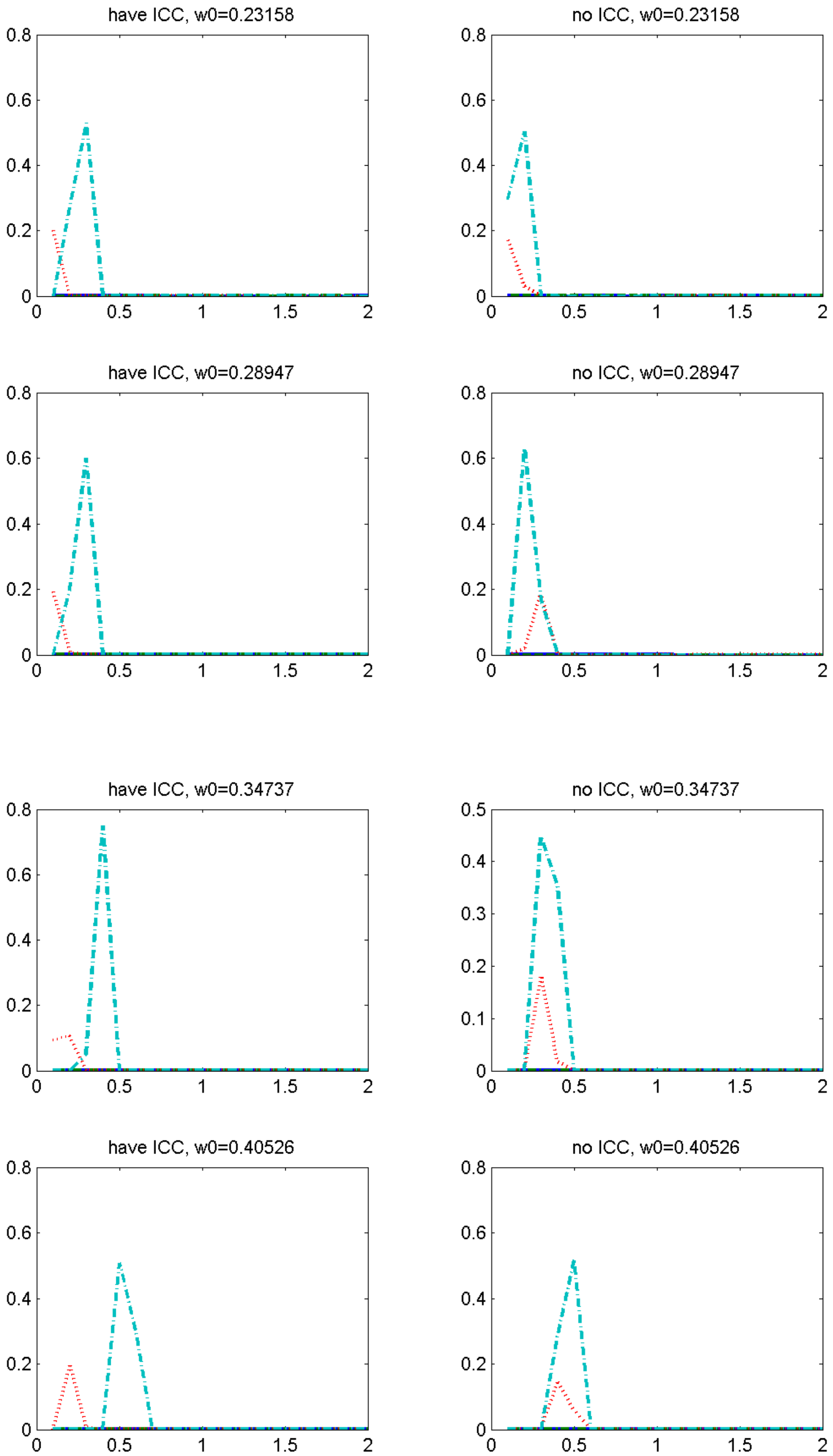
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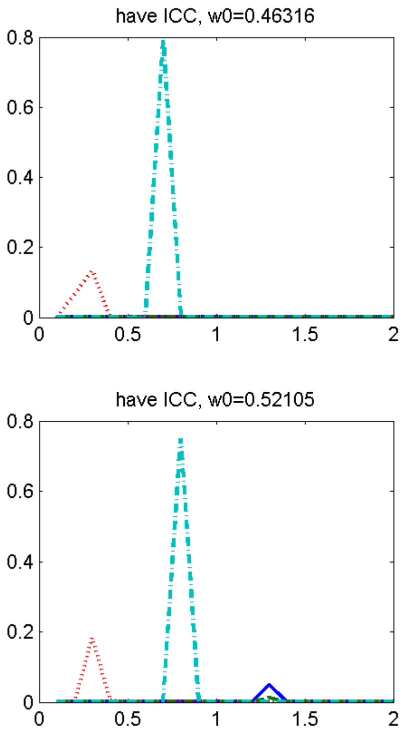
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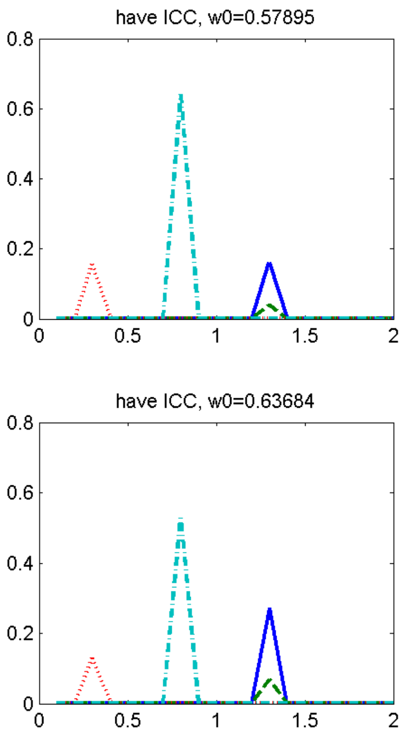
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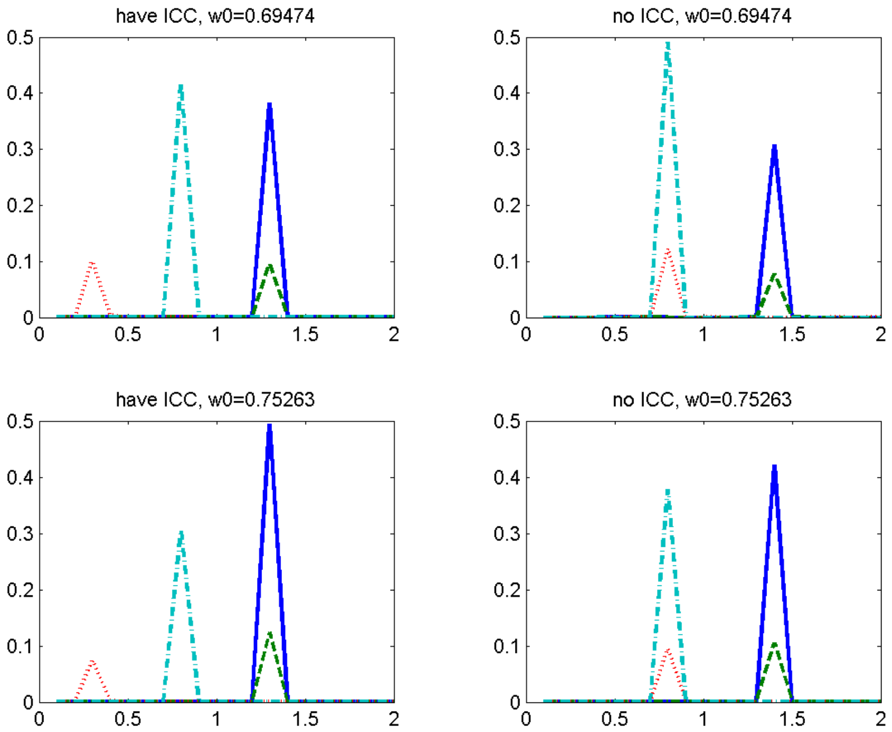
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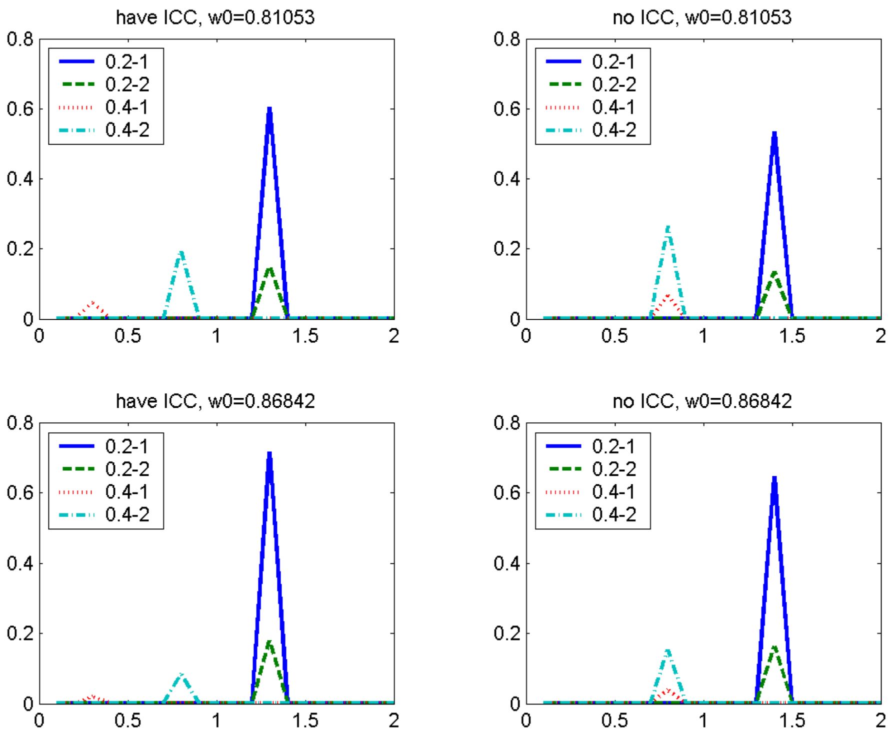
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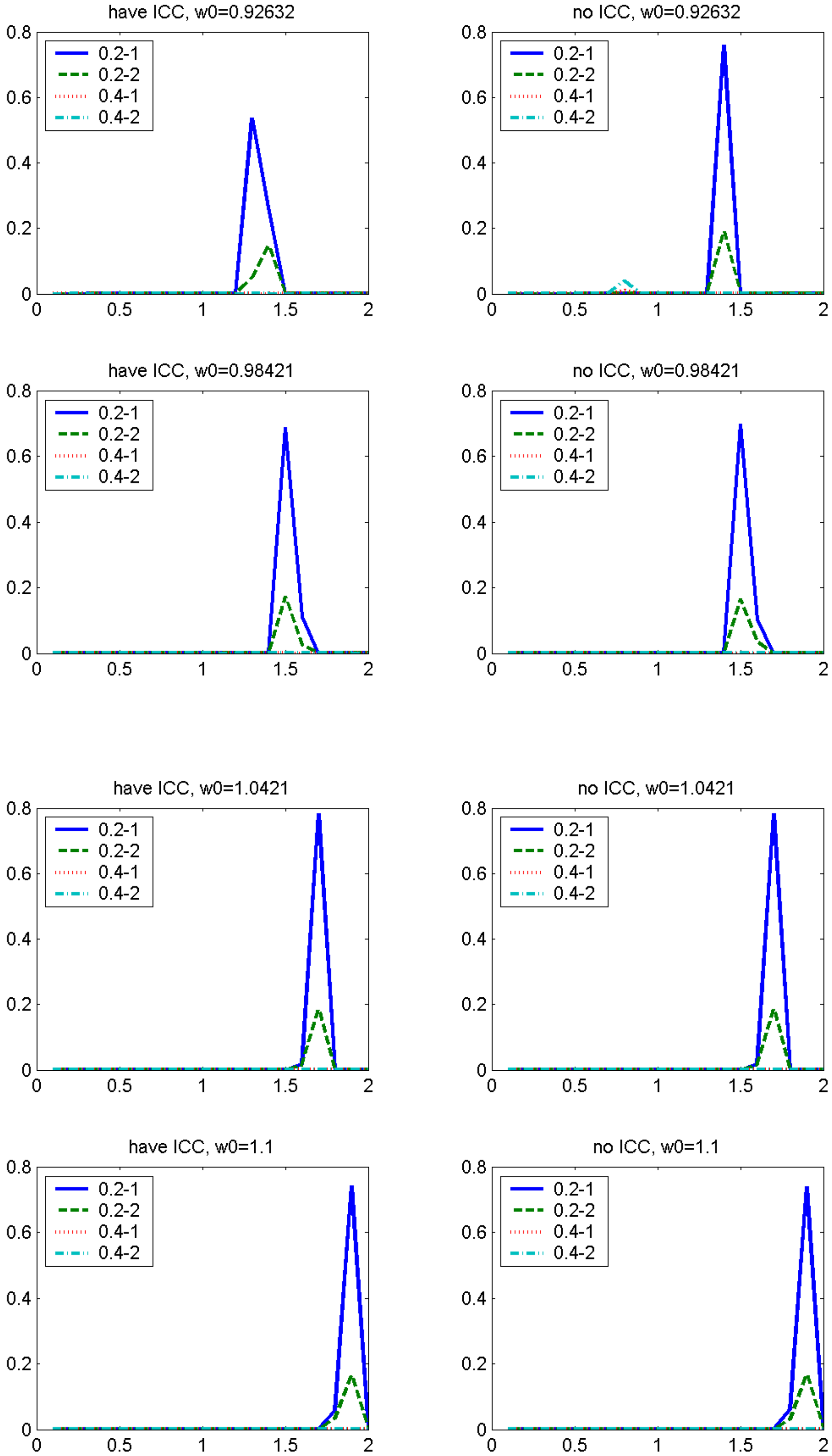
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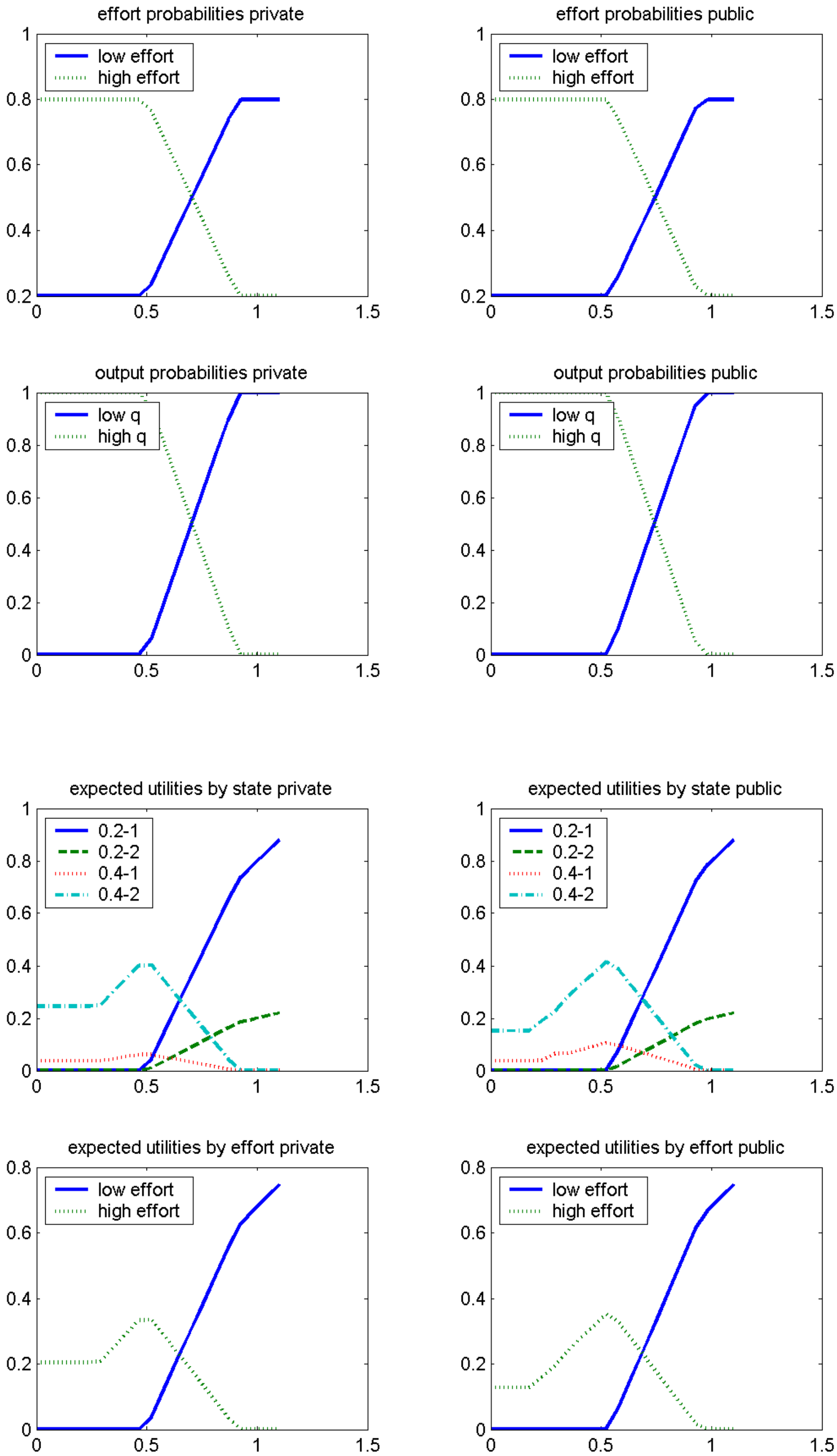
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low effort
high effort
effort probabilities public
low effort
high effort
output probabilities private
low q
high q
output probabilities public
low q
high q
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0.2-1
0.2-2
0.4-1
0.4-2
expected utilities by state public
0.2-1
0.2-2
0.4-1
0.4-2
expected utilities by effort private
low effort
high effort
expected utilities by effort public
low effort
high effort