Econ202B Homework #1 Anton Cheremukhin

January 25, 2006

Exercise 1 *Stokey-Lucas problems*

5.1

(a) Zero is always attainable, hence $\Gamma(x)$ is non-empty. (**4.1 holds**)

U is strictly increasing, hence, $F(x_t, x_{t+1}) = U(f(x_t) - x_{t+1}) \geq U(0)$ is bounded below. Additionally $0 < \beta < 1$, hence, $\lim \Sigma \beta^t F(x_t, x_{t+1})$ exists (**4.2 holds**).

(b) $X = [0, \overline{x}]$ is convex. $\Gamma : x \rightarrow [0, f(x)]$ is nonempty (by 4.1), compact-valued and continuous[1], since $f(x)$ is continuous and strictly increasing (**4.3 holds**).

[1] ►*) Choose x. $0 \in \Gamma(x) \Rightarrow$ non-empty.Choose $y \in \Gamma(x)$ and $x_n \rightarrow x$. Let $y_n = \gamma f(x_n), \gamma = y/f(x) \leq 1$. Then $y_n \in \Gamma(x)$ and $\lim y_n = \gamma \lim f(x_n) = \gamma f(x) = y$ by continuity of f. $\Rightarrow$ Γ - l.h.c.

**) Given x, $[0, f(x)]$ is compact and therefore $\Gamma(x)$ is compact-valued. Take any $x_n \rightarrow x$ and $y_n \in \Gamma(x_n)$. Define $\varepsilon = \sup_{x_n} \|x_n - x\|$, let $N(x, \varepsilon)$ a be closed ε-neighborhood and $\overline{f} = \max_{x' \in N(x,\varepsilon)} f(x')$. Since $z \in \left[0, \overline{f}\right]$ is compact, $\exists y_{n_k} \in y_n$ convergent to y. Since $y_{n_k} \leq f(x_{n_k})$ by continuity of f is must be that $y \leq f(x)$. $\Rightarrow$ Γ - u.h.c.

***) Γ - l.h.c. and u.h.c. $\Rightarrow$ Γ - continuous.◄

Since x_{t+1} has to be smaller than $f(x_t)$ and $x > f(x)$ for all $x > \overline{x}$, x_{t+1} and x_t are in the interval $[0, \overline{x}]$. Since both U and f are also continuous, F is bounded and continuous. $0 < \beta < 1$ is given. (**4.4 holds**)

Since both U and f are continuous and strictly increasing, F is strictly increasing in its first argument (**4.5 holds**)

If $x \leq x\prime$ then $f(x) \leq f(x\prime)$. Hence $\Gamma(x)$ is monotone (**4.6 holds**).

(c) $f(x)$ is concave, hence $f(x) - y$ is concave. U is strictly concave, hence F is strictly concave (**4.7 holds**).

$y \leq f(x)$, $y' \leq f(x')$, $\theta y + (1-\theta)y' \leq \theta f(x) + (1-\theta) f(x') \leq f(\theta x + (1-\theta)x')$

Hence, $\Gamma(x)$ is convex (**4.8 holds**).

5.5

(a) $v(k) = \max\{k, \beta v(h(k))\}$.

Assumptions:

A1: Set $\mathbf{h}(0) \geq \mathbf{0}$ because the tree is growing.

A2: Let the **state space be bounded**: $\left[0, \overline{k}\right]$ where $\overline{k}$ is big.

A3: Let **h(k) be increasing**,

A4: **h(k) crosses k/β only once**: **from above (except possibly zero)**.

i.e. $\exists k_0$: $\forall k > k_0 : h(k) < k/\beta$. $\forall$ and $\forall k < k_0 : h(k) > k/\beta$.

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Blackwell conditions hold:

B1) If $f(x) \leq g(x)$ for all x, then $\beta f(h(k)) \leq \beta g(h(k))$ and $\max\{k, \beta f(h(k))\} \leq \max\{k, \beta g(h(k)\}$

B2) $\max\{k, \beta[f(h(k)) + a]\} \leq \max\{k + \beta a, \beta[f(h(k)) + a]\} = \max\{k, [f(h(k))]\} + \beta a$

Hence, there exists a unique value function $v(.)$ (fixed point).

$v(.)$ is increasing: If $f(k') \leq f(k'')$ for all $k' \leq k''$, then $\beta f(h(k')) \leq \beta f(h(k''))$ and $\max\{k, \beta f(h(k'))\} \leq \max\{k, \beta f(h(k''))\}$. Hence, limiting $v(.)$ is also increasing.

Proof of existence of a simple rule:

(1) Assume $\forall k > k_0 : f(k) \leq k$. Then $\max\{k, \beta[f(h(k))]\} \leq \max\{k, \beta f(k/\beta)\} = \beta \max\{k/\beta, f(k/\beta)\} = \beta\{k/\beta\} = k$, which means that once we start with some function f satisfying the assumption all the future iterations also stick to it. So does the fixed point. Hence, $\forall k > k_0 : v(k) \leq k$. Since $v(k) \geq k$ by definition, we can state that $\forall k > k_0 : v(k) = k$. Hence, you should cut down the tree for all $k > k_0$.

(2) Assume $\forall k < k_0 : f(k) > k$. Then $\max\{k, \beta[f(h(k))]\} \geq \max\{k, \beta f(k/\beta)\} = \beta \max\{k/\beta, f(k/\beta)\} = \beta f(k/\beta) > \beta k/\beta = k$, which means that once we start with some function f satisfying the assumption all the future iterations also stick to it. So does the fixed point. Hence, $\forall k < k_0 : v(k) > k$. Hence, you should not cut down the tree for all $k < k_0$.

So, we get a simple **decision rule**: cut the tree if $k \geq k_0 : h(k_0) = k_0/\beta$.

Notice that we didn't mention continuity or concavity. If the function is concave there cannot be more than one crossing. However, there could be no crossing at all in which case we should either never cut the tree or cut it anyway.

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(b) $v(k) = \max\{k - c + \beta v(h(0)), \beta v(h(k)), k\}$

Assumptions:

A1: Set $\mathbf{h}(0) \geq \mathbf{c}/\beta > \mathbf{0}$ because the tree is growing from zero-size.

A2: Let the **state space be bounded**: $[0, \overline{k}]$ where $\overline{k}$ is big.

A3: Let $\mathbf{h(k)}$ **be increasing**.

A4: $\mathbf{h(k) - h(0)}$ **crosses** $(\mathbf{k - c})/\beta$ **only once**: **from above (except possibly zero)**.

i.e. $\exists k_0$: $\forall k > k_0 : h(k) < h(0) + (k - c)/\beta$. $\forall$ and $\forall k < k_0 : h(k) > h(0) + (k - c)/\beta$.

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The problem satisfies Blackwell conditions and there exists a unique fixed point:

B1) If $f(x) \leq g(x)$ for all x, then $\beta f(h(k)) \leq \beta g(h(k))$ and
$\max\{k - c + \beta f(h(0)), \beta f(h(k)), k\} \leq \max\{k - c + \beta g(h(0)), \beta g(h(k), k\}$

B2) $\max\{k - c + \beta[f(h(0) + a]), \beta[f(h(k)) + a], k\} =$
$\beta a + \max\{k - c + \beta f(h(0)), \beta f(h(k)), k - \beta a\} \leq \max\{k - c + \beta f(h(0)), \beta f(h(k)), k\} + \beta a$

$v(.)$ is increasing: If $f(k') \leq f(k'')$ for all $k' \leq k''$, then $\beta f(h(k')) \leq \beta f(h(k''))$ and
$\max\{k' - c + \beta f(h(0)), \beta f(h(k')), k'\} \leq \max\{k'' - c + \beta f(h(0)), \beta f(h(k'')), k''\}$.

Given that the initial guess is increasing, the limiting $v(.)$ is also increasing.

Using A1 we show that $v(h(0)) \geq h(0) \geq c/\beta$:

Take some f such that $f(h(0)) \geq h(0) \geq c/\beta$. Then
$\max\{h(0) - c + \beta f(h(0)), \beta f(h(h(0))), h(0)\} \geq \max\{h(0) - c + \beta f(h(0)), \beta f(h(0)), h(0)\}$
$\geq \max\{h(0) - c + \beta h(0), \beta h(0), h(0)\} \geq h(0) \geq c/\beta$

Hence, by induction $v(h(0)) \geq h(0) \geq c/\beta$ and $[-c + \beta v(h(0))]$ is a non-negative constant.

Therefore, it is always better to replant the tree and the third option is dominated:

$$\boxed{v(k) = \max\{k - c + \beta v(h(0)), \beta v(h(k))\}}$$

Proof of existence of a simple rule:

(1) Suppose $\forall k > k_0 : f(k) \leq \beta f(h(0)) - c + k$. Then
$Tf(k) = \max\{k - c + \beta f(h(0)), \beta f(h(k))\} \leq \max\{k - c + \beta f(h(0)), \beta(\beta f(h(0)) - c + h(k))\}$
$\leq \max\{k - c + \beta f(h(0)), \beta(\beta f(h(0)) - c) + \beta h(0) - c + k)\} \leq$

$k-c+\beta f\left(h(0)\right)+(1-\beta)\,\beta\max\left\{0,h(0)-f(h(0))\right\}=k-c+\beta f\left(h(0)\right)$, which means that once we start with some function f satisfying the assumption all the future iterations also stick to it. So does the fixed point. Hence, $\forall k>k_0: v(k)\leq\beta v\left(h(0)\right)-c+k$. Since $v(k)\geq\beta v\left(h(0)\right)-c+k$ by definition, we can state that $\forall k>k_0: v(k)=\beta v\left(h(0)\right)-c+k$. Hence, you should cut down and replant for all $k>k_0$.

(2) Suppose now that $\forall k<k_0: f(k)>\beta f\left(h(0)\right)-c+k$. Then

$\max\left\{k-c+\beta f\left(h\left(0\right)\right),\beta f\left(h\left(k\right)\right)\right\}\geq\max\left\{k-c+\beta f\left(h\left(0\right)\right),\beta(\beta f\left(h(0)\right)-c+h(k))\right\}>$
$\max\left\{k-c+\beta f\left(h\left(0\right)\right),\beta(\beta f\left(h(0)\right)-c+h(0)+\left(k-c\right)/\beta)\right\}=$
$k-c+\beta f\left(h\left(0\right)\right)+\beta\max\left\{0,h(0)-c\right\}=k-c+\beta f\left(h\left(0\right)\right)$, which means that once we start with some function f satisfying the assumption all the future iterations also stick to it. So does the fixed point. Hence, $\forall k<k_0: v(k)>k-c+\beta f\left(h\left(0\right)\right)$. Hence, you should not cut down the tree for all $k<k_0$.

So, we get a simple **decision rule**: cut the tree if $k\geq k_0: h(k_0)=h(0)+\left(k_0-c\right)/\beta$.

Again we didn't mention continuity or concavity. If the function is concave there cannot be more than one crossing. However, there could be no crossing at all in which case we should either never cut the tree or cut it every period.

Exercise 2 *Monk problem*

$$v\left(k\right)=\sup_{k'\leq k/\beta}\left\{-k+\beta k'+\beta v\left(k'\right)\right\}$$

The problem as stated has a continuum of solutions: $v(k)=\theta k$ where $\theta\geq-1$:

$$v\left(k\right)=\sup_{k'\leq k/\beta}\left\{-k+\beta k'+\beta\theta k'\right\}=\sup_{k'\leq k/\beta}\left\{-k+\beta(1+\theta)k'\right\}=-k+\beta(1+\theta)\tfrac{k}{\beta}=\theta k$$

(a) The Blackwell conditions do hold, but since the state space $X=R$ is unbounded the value functions are unbounded on X unless $\theta=0$.

(b) If we constrain the state space to be $X=\left[0,\overline{k}\right]$ then the problem has a unique solution: $v(k)=-k$. In this case the individual has to consume everything in finite time.

Another way to solve the problem is to restrict ourselves to functions bounded on R. Then $v(k)=0$ is the unique solution. In this case the man can infinitely accumulate capital.

Exercise 3 *Two-sector economy*

(1) $\max\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\sigma}}{1-\sigma}$ s.t. $c_t\leq l_{ct}^{\alpha}k_{ct}^{1-\alpha},\qquad i_t\leq l_{it}^{\gamma}k_{it}^{1-\gamma},$
$l_{it}+l_{ct}\leq1,\qquad k_{it}+k_{ct}\leq k_t,\qquad k_{t+1}=(1-\delta)k_t+i_t,\qquad k_0$ given

k_t is the state variable. k_{ct} and l_{ct} are the control variables.

Recursive formulation:

$$v\left(k\right)=\max_{0\leq k_c\leq k,0\leq l_c\leq1}\left\{\frac{\left(l_c^{\alpha}k_c^{1-\alpha}\right)^{1-\sigma}}{1-\sigma}+\beta v\left[(1-\delta)k+(1-l_c)^{\gamma}(k-k_c)^{1-\gamma}\right]\right\}$$

This operator satisfies the Blackwell conditions (define state space for k as $\left[0,\overline{k}\right]$):

If $v'\leq v''$ on the whole state space then

$$Tv'\left(k\right)=\max_{0\leq k_c\leq k,0\leq l_c\leq1}\left\{\begin{array}{c}\frac{\left(l_c^{\alpha}k_c^{1-\alpha}\right)^{1-\sigma}}{1-\sigma}+\beta v''\left[(1-\delta)k+(1-l_c)^{\gamma}(k-k_c)^{1-\gamma}\right]+\\\beta(v'\left[(1-\delta)k+(1-l_c)^{\gamma}(k-k_c)^{1-\gamma}\right]-\\v''\left[(1-\delta)k+(1-l_c)^{\gamma}(k-k_c)^{1-\gamma}\right])\end{array}\right\}\leq$$

$$\max_{0\leq k_c\leq k,0\leq l_c\leq1}\left\{\frac{\left(l_c^{\alpha}k_c^{1-\alpha}\right)^{1-\sigma}}{1-\sigma}+\beta v''\left[(1-\delta)k+(1-l_c)^{\gamma}(k-k_c)^{1-\gamma}\right]\right\}+$$
$$\max_{0\leq k_c\leq k,0\leq l_c\leq1}\beta(v'\left[(1-\delta)k+(1-l_c)^{\gamma}(k-k_c)^{1-\gamma}\right]-v''\left[(1-\delta)k+(1-l_c)^{\gamma}(k-k_c)^{1-\gamma}\right])\leq Tv''\left(k\right)$$

$T(v+a)(k) = \max_{0\leq k_c\leq k, 0\leq l_c\leq 1}\left\{\frac{\left(l_c^\alpha k_c^{1-\alpha}\right)^{1-\sigma}}{1-\sigma} + \beta(v\left[(1-\delta)k + (1-l_c)^\gamma (k-k_c)^{1-\gamma}\right] + a)\right\} = Tv(k) + \beta a$

First Order Conditions:

$(1-\alpha)\, l_c^\alpha k_c^{-\alpha}\left(l_c^\alpha k_c^{1-\alpha}\right)^{-\sigma} = \beta v'\left[(1-\delta)k + (1-l_c)^\gamma (k-k_c)^{1-\gamma}\right](1-\gamma)(1-l_c)^\gamma (k-k_c)^{-\gamma}$

"marginal utility of consumption share of capital is equal to its marginal loss in lifetime utility"

$\alpha l_c^{\alpha-1} k_c^{1-\alpha}\left(l_c^\alpha k_c^{1-\alpha}\right)^{-\sigma} = \beta v'\left[(1-\delta)k + (1-l_c)^\gamma (k-k_c)^{1-\gamma}\right]\gamma(1-l_c)^{\gamma-1}(k-k_c)^{1-\gamma}$

"marginal utility of consumption share of labor is equal to its marginal loss in lifetime utility"

Envelope Theorem:

$v'(k) = \beta v'\left[(1-\delta)k + (1-l_c)^\gamma (k-k_c)^{1-\gamma}\right]\left\{(1-\delta) + (1-\gamma)(1-l_c)^\gamma (k-k_c)^{-\gamma}\right\}$

"marginal lifetime utility is equal to marginal instantaneous utility on the optimal plan"

Combining the three equations with $k = (1-\delta)\,k + (1-l_c)^\gamma (k-k_c)^{1-\gamma}$ we get a system of 4 equations with 4 unknowns. Solve them for k: $\frac{\gamma(1-\alpha)l_c}{\alpha(1-\gamma)k_c} = \frac{1-l_c}{k-k_c} \quad\Rightarrow\quad \frac{l_c}{k_c} = \frac{\alpha(1-\gamma)}{(1-\alpha)\gamma k+(\alpha-\gamma)k_c}$,

$\frac{\frac{1}{\beta}-(1-\delta)}{1-\gamma} = \left(\frac{\gamma(1-\alpha)}{(1-\alpha)\gamma k+(\alpha-\gamma)k_c}\right)^\gamma \quad\Rightarrow\quad k_c = \frac{\gamma\delta+\frac{1}{\beta}-1}{\delta+\frac{1}{\beta}-1}k \quad\Rightarrow\quad k^* = \frac{\gamma(1-\alpha)\left(\frac{\frac{1}{\beta}-(1-\delta)}{1-\gamma}\right)^{-1/\gamma}}{\left((1-\alpha)\gamma+(\alpha-\gamma)\frac{\gamma\delta+\frac{1}{\beta}-1}{\delta+\frac{1}{\beta}-1}\right)}$

(2) $\max\sum_{t=0}^\infty \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$ s.t. $c_t \leq l_{ct}^\alpha k_{ct}^{1-\alpha}, \quad i_{it} + i_{ct} \leq l_{it}^\gamma k_{it}^{1-\gamma},$

$l_{it} + l_{ct} \leq 1, \quad k_{it+1} \leq (1-\delta)\,k_{it} + i_{it}, \quad k_{ct+1} = (1-\delta)k_{ct} + i_{ct}, \quad k_{c0}, k_{i0}$ given

k_{ct} and k_{it} are the state variables. k_{ct+1} and k_{it+1} are the control variables.

Recursive formulation:

$i_{it} = (1-\delta)\,k_{it} - k_{it+1} \qquad i_{ct} = (1-\delta)k_{ct} - k_{ct+1} \qquad l_{it}^\gamma = \frac{(1-\delta)(k_{it}+k_{ct})-(k_{it+1}+k_{ct+1})}{k_{it}^{1-\gamma}}$

$l_{ct} = 1 - \frac{(1-\delta)(k_{it}+k_{ct})-(k_{it+1}+k_{ct+1})}{k_{it}^{1-\gamma}} \qquad c_t = k_{ct}^{1-\alpha}\left(\frac{(1-\delta)(k_{it}+k_{ct})-(k_{it+1}+k_{ct+1})}{k_{it}^{1-\gamma}}\right)^\alpha$

$v(k_i, k_c) = \max_{0\leq k_c'\leq \overline{k_c}, 0\leq k_i'\leq \overline{k_i}}\left\{\frac{\left(k_c^{1-\alpha}\left(\frac{(1-\delta)(k_i+k_c)-(k_i'+k_c')}{k_i^{1-\gamma}}\right)^\alpha\right)^{1-\sigma}}{1-\sigma} + \beta v\left[k_i', k_c'\right]\right\}$

Exercise 4 *Numeric Example*

$\max\sum_{t=0}^\infty \beta^t \log c_t$ s.t. $c_t + k_{t+1} \leq Ak_t^\alpha + (1-\delta)\,k_t, \quad k_0$ given

$\beta = 0.6 \qquad A = 15 \qquad \alpha = 0.3 \qquad \delta = 0.5$

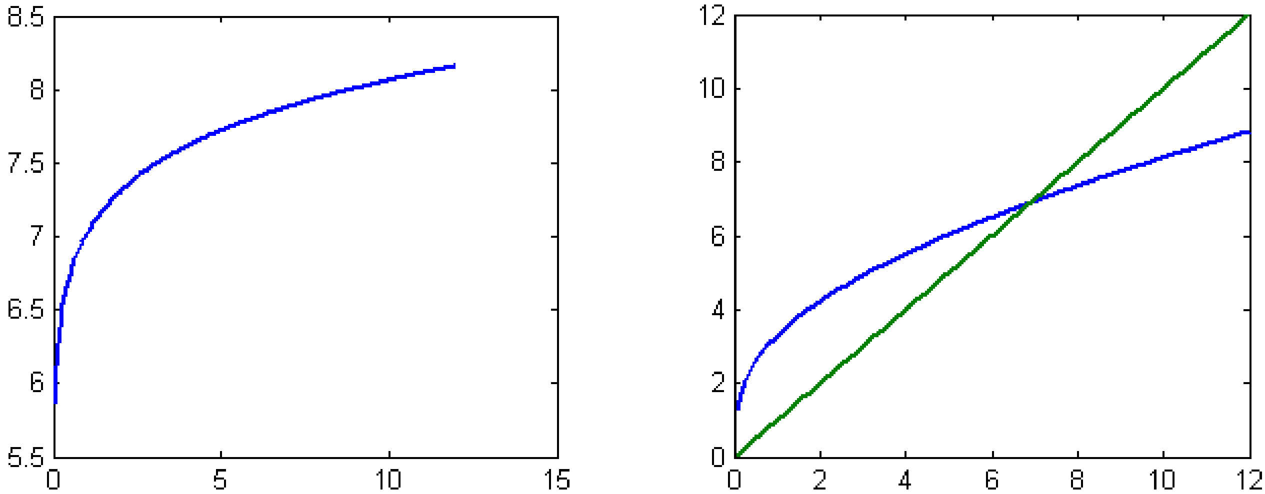
$v(k) = \max_{0\leq k'\leq \overline{k}}\left\{\log\left(Ak^\alpha + (1-\delta)\,k - k'\right) + \beta v(k')\right\}$

Numerival computations give the steady-state value around 7.

Check using FOC, ENV and $k = k'$:

$\frac{1}{Ak^\alpha+(1-\delta)k-k'} = \beta v'(k') \qquad v'(k) = \frac{A\alpha k^{\alpha-1}+(1-\delta)}{Ak^\alpha+(1-\delta)k-k'} \qquad k^* = \left(\frac{\alpha\beta A}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}} = 6.87$

Value Function and Policy Function



Code:

```
clear; beta=0.6; alpha=0.3; A=15; delta=0.5;        % parameters from the task
epsilon = 0.00001;       % parameter for computations used for convergence detection
% optimal steady state value, k*
k_opt = ((1/beta-(1-delta))/(A*alpha))^(1/(alpha-1))
% define main grid as an array of values from 0.05 to 12
K_min = 0.05;            K_max = 12;            K_step = 0.05;
K_grid = [K_min:K_step:K_max];
K_finestep = K_step/20;             % interpolation grid
K_finegrid = [K_min:K_finestep:K_max];
% value function iteration algorithm, computing arrays of V_new for all k from K_grid
V_new = ones(size(K_grid));        V_old = zeros(size(K_grid));
K_pf = zeros(size(K_grid));
while (abs(max(V_new - V_old)) > epsilon)
        V_old = V_new;
        for n = 1 : size(K_grid,2)
              k = K_min + K_step*(n-1);
              % Calculate maximum over all k_primes from our supergrid.
              [V_new(n),K_pf(n)] = max(log(max(0.000000001*ones(size(K_finegrid)),
ones(size(K_finegrid))*(A*(k^alpha)+(1-delta)*k) - K_finegrid)) +
beta*interp1(K_grid, V_old, K_finegrid, 'spline'));
        end
end
K_pf = K_finegrid(K_pf);
figure(1)
plot(K_grid,V_new,...
                        'LineWidth',2)
figure(2)
plot(K_grid,K_pf,K_grid,K_grid,...
                        'LineWidth',2)
```