Problem Set 3

1) Consider the following economic model. A representative agent produces a single commodity y_t from capital k_t using the technology

1.1) $$y_t = s_t k_t^{\alpha}, \quad 0 < \alpha < 1 .$$

s_t is an i.i.d. random variable defined on the probability space (S, B, λ) where $S = [s_1, s_2] \subset R_+$, B is the Borel sigma-algebra and

1.2) $$\int s d\lambda(s) = 1.$$

Output y_t can be consumed (consumption is denoted c_t) or invested in new capital, k_{t+1}. There is 100% depreciation, hence next period's capital stock k_{t+1} is defined by the expression

1.3) $$k_{t+1} = y_t - c_t, \quad t = 1, \ldots .$$

A history of shocks is a sequence $s^t \in S^t$ where the $k'th$ element of s^t for $k \le t$ is the shock that occurred at date k. The agent chooses a contingent plan $\left\{c\left(s^t\right), k_{t+1}\left(s^t\right)\right\}_{t=1}^{\infty}$ to maximize expected utility given by the expression

1.4) $$V = \sum_{t=1}^{\infty} E_1 \left\{\beta^{t-1} \log\left(c\left(s^t\right)\right)\right\}$$

subject to the constraints (1.3) and the initial stock of capital k_1.

1.A) Conjecture that there is a solution of the form

1.5) $$c_t = \lambda y_t$$

and find an expression for the value of λ in terms of the parameters. HINT: try to write the Euler equation as a difference equation in $\left(\frac{k_{t+1}}{c_t}\right)$ using equations 1.1 and 1.3.

1.B) Prove that this conjectured solution also satisfies the transversality condition. (This is the same condition as in the non-stochastic model with $u_c(c)$ replaced by $E\left\{u_c(c) \mid s_0\right\}$.

1.C) Find an explicit expression for the function $g(\bullet)$ where

1.6) $$k_{t+1} = g\left(k_t, s_t; \alpha, \beta\right)$$

describes the evolution of k_t in the optimal solution.

1.D) Let s_t be lognormally distributed. Let (X, B) be a measurable space where $X = R_+$ and let B be the Borel sigma-algebra. Find an explicit expression for the Markov transition function

1.7) $$P\left(\log\left(k_t\right), A\right)$$

associated with the stochastic difference equation 2.6) where A is an interval $[A_1, A_2]$ in R_+ and $P(\log(k_t), A)$ is the probability that $\log(k_{t+1})$ will be in A given $\log(k_t)$.

2) Let M be a family of probability measures defined on (X, χ) and let $\mu_0(A) \in \mathrm{M}$ be the probability at date 0 that $k_0 \in A$. Suppose further that μ_0 is the measure that puts probability 1 on the event $k = k_0$ and zero elsewhere and let f be a bounded function $X \to R$. Explain in words what is meant by each of the following expressions

2.i) $\int P(x, A)\, d\mu(x)$

2.ii) $\int P^n(x, A)\mu_0(dx)$

2.iii) $\int f(x')P(x, dx')$

2.iv) $\int f(x')P^n(x, dx')$

3) Consider the stochastic difference equation

3.1) $$x_{t+1} = g(x_t, s_t) = 1 - \frac{a}{x_t} + s_t.$$

Let (R_+, B) be a measurable space and (S, B, μ) be a measure space where B is the Borel sigma algebra and S = $[-e, e] \subset R_+$. Let μ be the uniform measure on $[-e, e]$ and restrict e to the open interval $0 < e < 1$ and let $\Lambda(R, B)$ be the set of probability measures on (R, B).

3.A) Draw graphs of x_{t+1} against x_t for $s = e$ and for $s = -e$.

3.B) Find a condition on the values of a and e such that the non-stochastic difference equation

3.2) $$x_{t+1} = g(x_t, s)$$

has at least two fixed points for all $s \in [-e, e]$. For the remaining parts of the question assume that this condition holds.

3.C) Imposing your condition from part C, find a point C such that if $x_0 > C$ then $x_t > C$ for all t.

3.D) Find an expression for the transition function $Q(x, A)$ for sets A of the form $A = [\alpha, \beta]$ where $0 < \alpha < \beta$.

3.E) Consider the Markov Process defined by the transition function $Q(x, A)$ restricted to the set $X = [C, \infty]$. Prove that for all $x_0 \in X$, $Q^T(x, A)$ converges to a unique probability measure μ^*.