Problem Set 1

1. Consider each of the following linear difference equations

$$y_t = 1 - 0.5 y_{t-1}, \quad y_0 = 1 \tag{1.1}$$
$$y_t = y_{t-1} + 3, \quad y_0 = 1 \tag{1.2}$$
$$y_t = 1 - y_{t-1}, \quad y_0 = 0.5 \tag{1.3}$$
$$y_t = 2 + 1.2 y_{t-1}, \quad \lim_{t\to\infty} |y_t| < \infty \tag{1.4}$$
$$y_t = 1 + (1/3)\, y_{t-1} \quad y_0 = 0 \tag{1.5}$$
$$y_t = 0.5 y_{t-1} + 0.2 \quad \lim_{t\to\infty} |y_t| < \infty \tag{1.6}$$

For each example

a) Find a sequence of real numbers $\{y_t\}_{t=0}^{\infty}$ that solves the equation.
b) Find the steady state of the difference equation (if it exists). If not, say so.
c) If a steady state exists is it stable or unstable?

2. Consider each of the following systems of linear difference equations

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix}, \quad y_0 = 1, \; x_0 = 1. \tag{1.7}$$
$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 \\ 0.8 & 0.2 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix}, \quad y_0 = 1, \; x_0 = 1. \tag{1.8}$$
$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 \\ 2.8 & 0.2 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} \quad x_0 = 1, \; y_0 = 1. \tag{1.9}$$
$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 \\ 2.8 & 0.2 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} \quad x_0 = 1, \quad \lim_{t\to\infty} |y_t| < \infty \tag{1.10}$$

For each example

a) Use Matlab to calculate the eigenvalues and eigenvectors of the system.
b) Find sequences of real numbers $\{x_t, y_t\}_{t=0}^{\infty}$ that solve the system.
c) Find the steady state of the difference equation (if it exists). If not, say so.
d) If a steady state exists is it stable or unstable?

3. Consider the following linear model:

$$y_t = \alpha y^e_{t+1} + \beta x_t + u_t \tag{1.11}$$
$$x_t = \gamma x_{t-1} + \delta + v_t \tag{1.12}$$

where Equation (1.11) describes the evolution of an endogenous variable and equation (1.12) represents a policy rule. The parameters α,β γ and δ are all positive and α and γ are both less than one. The terms u_t and v_t are independent, serially uncorrelated error terms with zero mean.

a) Under the assumption that the subjective expectation, y^e is determined adaptively by the rule:

$$y_{t+1}^e = \lambda y_t^e + (1-\lambda) y_t \tag{1.13}$$

where, $0 < \lambda < 1$, find a stochastic difference equation involving only the observable variables y_t and x_t (assume that y_t^e is *not* observable), lags of y_t and x_t and (lags of) the error term u_t that describes the behavior of y_t through time.

b) Assume that the support of u_t and v_t is the interval [-a,a] where a is finite. Find the support of the distribution of x_t as $t \to \infty$. Find the support of the distribution of y_t (i.e. what are the largest and smallest values that x and y can attain in the limiting distribution.)

c) Find the unique rational expectations equilibrium of the above model, (i.e. replace Equation (1.13) with the rational expectations assumption).

4. Consider the following linear rational expectations model

$$y_t = \alpha E_t\left[y_{t+1}\right] + \delta + \psi y_{t-1} + u_t . \tag{1.14}$$

a) Find the characteristic roots (eigenvalues) of Equation (1.14) as functions of the parameters α and ψ.

b) Under what conditions on ψ and α will the roots of this equation be real?

c) If $\alpha = 1/2$ and $\psi = 3/8$ find the eigenvalues and the eigenvectors of this equation.

d) Find a stochastic difference equation that characterizes the unique rational expectations equilibrium.

5. For the following problem you may use Matlab or an equivalent programming language. Consider the following linear model

$$p_t = 0.4x_t + 0.2E_t\left[p_{t+1}\right] + 1 + u_t , \tag{1.15}$$

$$x_t = 0.2p_t + 0.5x_{t-1} + 1 + v_t, \quad x_0 = \overline{x}_0 . \tag{1.16}$$

Assume that u_t and v_t are i.i.d. random variables with mean zero.

Show that this model has a unique rational expectations equilibrium and find expressions for p_t and x_t as linear functions of u_t, v_t and x_{t-1}.