Economics 202A, Homework #4 Anton Cheremukhin

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Exercise 1 $U^A = \log(x_1x_2x_3) \qquad U^B = x_1x_2x_3 \qquad U^C = -(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3})$

$w^A = (1,0,0) \qquad w^B = (0,1,0) \qquad w^C = (0,0,1)$

(a) A,B: $p_1x_1 = p_ix_i \quad \Rightarrow \quad 3p_ix_i = p_{A,B}$

C: $\sqrt{p_1}x_1 = \sqrt{p_i}x_i \quad \Rightarrow \quad \sqrt{\frac{p_i}{p_1}}p_1x_1 = p_ix_i \quad \Rightarrow \quad p_1x_1(1+\sqrt{\frac{p_2}{p_1}}+\sqrt{\frac{p_3}{p_1}}) = p_3$

Demands: $x_i^A = \frac{p_1}{3p_i} \qquad x_i^B = \frac{p_2}{3p_i} \qquad x_i^C = \frac{p_3}{\sqrt{p_i}(\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3})}$ Excess demands:

$$f^A = x^A - w^A = \begin{bmatrix} -\frac{2}{3} \\ \frac{p_1}{3p_2} \\ \frac{p_1}{3p_3} \end{bmatrix} \quad f^B = x^B - w^B = \begin{bmatrix} \frac{p_2}{3p_1} \\ -\frac{2}{3} \\ \frac{p_1}{3p_3} \end{bmatrix} \quad f^C = x^C - w^C = \begin{bmatrix} \frac{p_3/\sqrt{p_1}}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}} \\ \frac{p_3/\sqrt{p_2}}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}} \\ -\frac{\sqrt{p_1}+\sqrt{p_2}}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}} \end{bmatrix}$$

(b) Aggregate excess demand $f(p) = f^A + f^B + f^C = \begin{bmatrix} \frac{1}{3}\frac{p_2}{p_1} + \frac{1}{\sqrt{p_1}}\frac{p_3}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}} - \frac{2}{3} \\ \frac{1}{3}\frac{p_1}{p_2} + \frac{1}{\sqrt{p_2}}\frac{p_3}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}} - \frac{2}{3} \\ \frac{1}{3}\frac{p_1}{p_3} + \frac{1}{3}\frac{p_2}{p_3} + \frac{-\sqrt{p_1}-\sqrt{p_2}}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}} \end{bmatrix}$

(c) $$f(\lambda p) = \begin{bmatrix} \frac{1}{3}\frac{\lambda p_2}{\lambda p_1} + \frac{1}{\sqrt{\lambda p_1}}\frac{\lambda p_3}{\sqrt{\lambda p_1}+\sqrt{\lambda p_2}+\sqrt{\lambda p_3}} - \frac{2}{3} \\ \frac{1}{3}\frac{\lambda p_1}{\lambda p_2} + \frac{1}{\sqrt{\lambda p_2}}\frac{\lambda p_3}{\sqrt{\lambda p_1}+\sqrt{\lambda p_2}+\sqrt{\lambda p_3}} - \frac{2}{3} \\ \frac{1}{3}\frac{\lambda p_1}{\lambda p_3} + \frac{1}{3}\frac{\lambda p_2}{\lambda p_3} + \frac{-\sqrt{\lambda p_1}-\sqrt{\lambda p_2}}{\sqrt{\lambda p_1}+\sqrt{\lambda p_2}+\sqrt{\lambda p_3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\frac{p_2}{p_1} + \frac{1}{\sqrt{p_1}}\frac{p_3}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}} - \frac{2}{3} \\ \frac{1}{3}\frac{p_1}{p_2} + \frac{1}{\sqrt{p_2}}\frac{p_3}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}} - \frac{2}{3} \\ \frac{1}{3}\frac{p_1}{p_3} + \frac{1}{3}\frac{p_2}{p_3} + \frac{-\sqrt{p_1}-\sqrt{p_2}}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}} \end{bmatrix} = f(p)$$

$p_1(\frac{1}{3}\frac{p_2}{p_1} + \frac{1}{\sqrt{p_1}}\frac{p_3}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}} - \frac{2}{3}) + p_2(\frac{1}{3}\frac{p_1}{p_2} + \frac{1}{\sqrt{p_2}}\frac{p_3}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}} - \frac{2}{3}) + p_3(\frac{1}{3}\frac{p_1}{p_3} + \frac{1}{3}\frac{p_2}{p_3} + \frac{-\sqrt{p_1}-\sqrt{p_2}}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}}) =$
$\frac{1}{3}p_2 + \frac{\sqrt{p_1}p_3}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}} - \frac{2}{3}p_1 + \frac{1}{3}p_1 + \frac{\sqrt{p_2}p_3}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}} - \frac{2}{3}p_2 + \frac{1}{3}p_1 + \frac{1}{3}p_2 + \frac{-\sqrt{p_1}p_3-\sqrt{p_2}p_3}{\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}} = 0$

Exercise 2 $\max[U^A + \log U^B + 2\sqrt{2}U^C] \qquad s.t. \sum(w^j - x^j) \le 0$

FOC: $\mu_j DU^j = q \quad \Rightarrow \begin{bmatrix} 1/x_1^A \\ 1/x_2^A \\ 1/x_3^A \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1/x_1^B \\ 1/x_2^B \\ 1/x_3^B \end{bmatrix} = 2\sqrt{2}\begin{bmatrix} (1/x_1^C)^2 \\ (1/x_2^C)^2 \\ (1/x_3^C)^2 \end{bmatrix}$

$2\sqrt{2}x_i^A = 2\sqrt{2}x_i^B = (x_i^C)^2 \quad \Rightarrow \quad$ BC: $\quad 2x + \sqrt[4]{8}\sqrt{x} = 1,$

Solution is: $x_i^A = x_i^B = \frac{1}{2} - \frac{1}{2}\sqrt[4]{8}\left(\frac{1}{4}\sqrt{\sqrt{8}+8} - \frac{1}{4}\sqrt[4]{8}\right) \simeq 0.161\,78 \qquad x_i^C = \sqrt[4]{8}\sqrt{x_i^A} \simeq 0.676\,44$

Transfers can be found as the difference between nominal values of incomes and endowments.

Normalized prices are: $\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1/x_1^B \\ 1/x_2^B \\ 1/x_3^B \end{bmatrix} = \begin{bmatrix} 6.181\,2 \\ 6.181\,2 \\ 1.478\,3 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \\ 0.239\,16 \end{bmatrix}$

$\tau_A = \tau_B = \begin{bmatrix} 1 & 1 & 0.239\,16 \end{bmatrix}\begin{bmatrix} 0.161\,78 - 1 \\ 0.161\,78 \\ 0.161\,78 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0.239\,16 \end{bmatrix}\begin{bmatrix} 0.161\,78 \\ 0.161\,78 - 1 \\ 0.161\,78 \end{bmatrix} = -0.637\,75$

$\tau_C = \begin{bmatrix} 1 & 1 & 0.239\,16 \end{bmatrix}\begin{bmatrix} 0.67644 \\ 0.67644 \\ 0.67644 - 1 \end{bmatrix} = 1.275\,5$

Exercise 3 *Using Walras Law:* $p_1f_1+p_2f_2+p_3f_3=0 \quad \Rightarrow \quad f_3=-\frac{p_1f_1+p_2f_2}{p_3}=-\frac{p_1}{p_2}-\frac{p_1(p_1-p_2)}{p_3p_3}$

Exercise 4

(i) $x_t^{it}+S_t=w_y^i \qquad x_{t+1}^{it}=w_o^i+R_tS_t \qquad \Rightarrow \qquad x_t^{it}+\frac{1}{R_t}x_{t+1}^{it}=w_y^i+\frac{1}{R_t}w_o^i$

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(ii) Demand: $\max[\log x_t^{it}+\beta\log x_{t+1}^{it}|x_t^{it}+\frac{1}{R_t}x_{t+1}^{it}=w_y^i+\frac{1}{R_t}w_o^i] \qquad \Rightarrow \qquad \frac{\beta}{x_{t+1}^{it}}=\frac{1}{x_t^{it}}\frac{1}{R_t}$

$\frac{1}{R_t}x_{t+1}^{it}=\beta x_t^{it} \qquad \Rightarrow \qquad x_t^{it}=\frac{1}{1+\beta}(w_y^i+\frac{1}{R_t}w_o^i), \qquad x_{t+1}^{it}=\frac{\beta}{1+\beta}R_t(w_y^i+\frac{1}{R_t}w_o^i)$

$f_i(R_t)=x_t^t-w_t^t=\frac{1}{1+\beta}(\frac{1}{R_t}w_o^i-\beta w_y^i)$.

If there is money, instead of loans then we have to redefine variables: $\pi_t=\frac{p_{t+1}}{p_t}=\frac{1}{R_t}$.

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(iii) A competitive equilibrium is a sequence of interest rates $\{R_t\}_{t=1}^{\infty}$ and allocations $\{x_t^{t-1},x_t^t\}_{t=1}^{\infty}$ such that 1) allocations solve agents' maximization problems given the interest rate and initial conditions: $x_t^{it}-w_y^i=f_i(R_t)$, $x_{t+1}^{it}-w_o^i=-R_tf_i(R_t)$, $x_1^{i0}=w^{i0}$ and 2) allocations are feasible in any period t: $x_t^{1t-1}-w_o^1+x_t^{1t}-w_y^1+x_t^{2t-1}-w_o^2+x_t^{2t}-w_y^2=0$.

Equilibrium could be found from: $-R_{t-1}f_1(R_{t-1})+f_1(R_t)-R_{t-1}f_2(R_{t-1})+f_2(R_t)=$

$=-R_{t-1}\frac{1}{1+\beta}(\frac{1}{R_{t-1}}w_o^1-\beta w_y^1)+\frac{1}{1+\beta}(\frac{1}{R_t}w_o^1-\beta w_y^1)-R_{t-1}\frac{1}{1+\beta}(\frac{1}{R_{t-1}}w_o^2-\beta w_y^2)+\frac{1}{1+\beta}(\frac{1}{R_t}w_o^2-\beta w_y^2)$

$=-R_{t-1}\frac{1}{1+\beta}(-\beta)+\frac{1}{1+\beta}(-\beta)-R_{t-1}\frac{1}{1+\beta}(\frac{1}{R_{t-1}})+\frac{1}{1+\beta}(\frac{1}{R_t})=0$

Assuming R constant we get: $(R-1)\frac{\beta}{1+\beta}=\frac{1}{1+\beta}\frac{R-1}{R}$.

Hence there are two solutions: $R=1$ (golden rule) , $R=\frac{1}{\beta}$ (autarky).

If $\beta>1$ then the golden rule is efficient, autarky is not and this is a Samuelson economy. If $\beta<1$ then autarky is the only (efficient) equilibrium and it is a Classical economy.

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(iv) In the golden rule equilibrium there is trade. Autarky is the absence of trade.

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(v) A competitive monetary equilibrium is a sequence of non-negative prices $\{p_t\}_{t=1}^{\infty}$ and allocations $\{x_t^{t-1},x_t^t\}_{t=1}^{\infty}$ such that 1) allocations solve agents' maximization problems given prices and initial conditions: $x_t^{it}-w_y^i=f_i(\frac{1}{\pi_t})$, $x_{t+1}^{it}-w_o^i=-\frac{1}{\pi_t}f_i(\frac{1}{\pi_t})$, $x_1^{i0}=w^{i0}+\frac{M}{p_1}$and 2) allocations are feasible in any period: $x_t^{1t-1}-w_o^1+x_t^{1t}-w_y^1+x_t^{2t-1}-w_o^2+x_t^{2t}-w_y^2=0$.

Using the definition above we similarly come to a condition:

$(1-\frac{1}{\pi_{t-1}})\frac{\beta}{1+\beta}=(\pi_t-1)\frac{1}{1+\beta} \qquad \Rightarrow \qquad \pi_t=(1-\frac{1}{\pi_{t-1}})\beta+1$

The initial value is given by $x_1^{i0}-w^{i0}=\frac{M}{p_1}$, hence

$\frac{M}{p_1}+x_1^{11}-w_y^1+\frac{M}{p_1}+x_1^{21}-w_y^2=\frac{2M}{p_1}+\frac{1}{1+\beta}(\pi_1w_o^1-\beta w_y^1)+\frac{1}{1+\beta}(\pi_1w_o^2-\beta w_y^2)=0$

$\frac{2M}{p_1}+\frac{1}{1+\beta}(-\beta)+\frac{1}{1+\beta}\pi_1=0 \qquad \Rightarrow \qquad \frac{2M}{p_1}=\frac{\beta-\pi_1}{1+\beta}$

So the equlibrium is defined by $\pi_t=(1-\frac{1}{\pi_{t-1}})\beta+1$ and initial condition $\frac{2M}{p_1}=\frac{\beta-\pi_1}{1+\beta}$.

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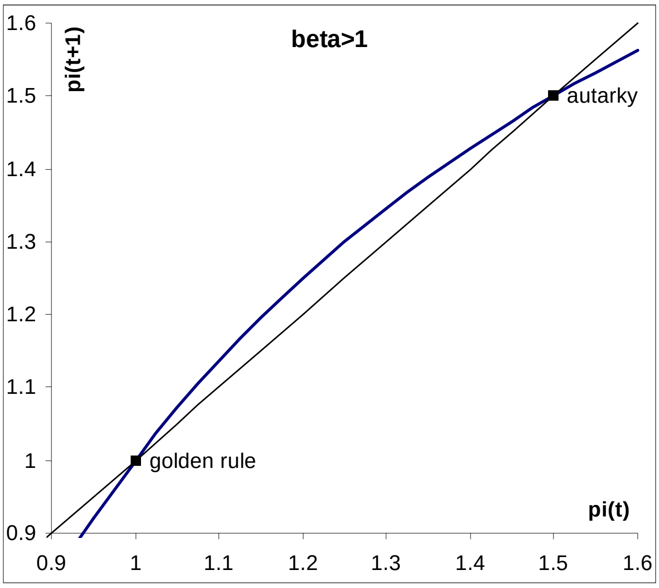
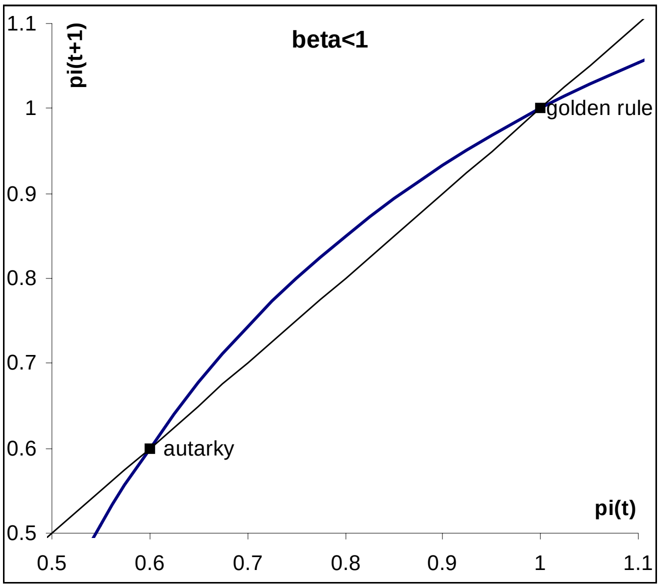
(vi) A stationary equilibrium is the one with allocations not changing over time, and hence $\frac{1}{\pi_t}=const$. The possible stationary solutions are $\pi=\beta$ and $\pi=1$. Plugging them into the initial condition we get: $\qquad \pi=1 \qquad \frac{2M}{p_1}=\frac{\beta-1}{\beta+1} \qquad\qquad \pi=\beta \qquad \frac{2M}{p_1}=0$

Case $\beta > 1$: From the picture it can be seen that there is a determinate golden rule equilibrium and there are multiple paths converging to the autarky equilibrium for any non-negative value of initial money stock (hence autarky is indeterminate). So, there are two stationary equilibria and a continuum of non-stationary equilibria, converging to one of them. The "golden rule" is the efficient one.



Case $\beta < 1$:The money stock needed to support the golden rule equilibrium is negative. We've got only one stationary equilibrium: the autarky one. It is determinate and efficient. All the other paths are impossible due to negative money stock and prices, needed to support them.



Actually, the only difference from part (iii) is that now the zero generation could escape from autarky using the money endowment.