Economics 202A, Homework #3 Anton Cheremukhin

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Exercise 1

$\max V = E_1 \left\{ \sum_{t=1}^{\infty} \beta^{t-1} \log(c(s_t)) \right\}, \qquad 0 < \beta < 1$
s.t. $k_{t+1} = y_t - c_t, \qquad t = \overline{1\infty}, \qquad K_1 = \bar{K}$
s.t. $Y_t = s_t k_t^{\alpha} \qquad 0 < \alpha < 1$
s.t. $(S, B, \lambda) \qquad S = [s_1, s_2] \subset R_+ \qquad B = Borel \qquad E[s] = \int s d\lambda(s) = 1$
Object of choice: $\{c(s^t), k_{t+1}(s^t)\}_{t=1}^{\infty}$, where $s^t = \{s_1, s_2 ... s_t\}$ - history of shocks.
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(a) Conjecture solution of the form: $c_t = \varphi y_t$. Then $\frac{c_t}{y_t} = \varphi$, $\frac{k_{t+1}}{y_t} = 1 - \varphi$.
$V = E_1 \left\{ \sum_{t=1}^{\infty} \beta^{t-1} \log(c(s_t)) \right\} = E_1 \left\{ \sum_{t=1}^{\infty} \beta^{t-1} \log(s_t k_t^{\alpha} - k_{t+1}(s_t)) \right\}$
FOC: $0 = E_t[-\frac{1}{c_t} + \beta s_{t+1} \alpha k_{t+1}^{\alpha-1} \frac{1}{c_{t+1}}] = E_t[-\frac{k_{t+1}}{c_t} + \beta\alpha \frac{y_{t+1}}{c_{t+1}}] = -\frac{k_{t+1}}{c_t} + \alpha\beta + \alpha\beta E_t[\frac{k_{t+2}}{c_{t+1}}]$

Inserting conjectured solution and assuming perfect-foresight we get:
$0 = -\frac{k_{t+1}}{y_t}\frac{y_t}{c_t} + \alpha\beta E_t[\frac{y_{t+1}}{c_{t+1}}] = -\frac{(1-\varphi)}{\varphi} + \frac{\alpha\beta}{\varphi} = \frac{\varphi + \beta\alpha - 1}{\varphi}$
Hence, $\varphi = 1 - \alpha\beta$.
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(b) Transversality condition holds: $\lim_{t\to\infty} E\{\beta^{t-1}\frac{k_{t+1}}{c_t}\} = \lim_{t\to\infty} \beta^{t-1}(\frac{1}{\varphi} - 1) = 0$.
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(c) $k_{t+1} = (1-\varphi) y_t = \alpha\beta s_t k_t^{\alpha} = g(k_t, s_t, \alpha, \beta)$
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(d) $\log s_t \sim N(0, \sigma^2)$
$\log k_{t+1} = c_0 + \log s_t + \alpha \log k_t$
$P(\log k_t, [A_1, A_2]) = \Pr[A_1 < \log k_{t+1} < A_2] = \Pr[A_1 < c_0 + \log s_t + \alpha \log k_t < A_2] =$
$\Pr[A_1 - c_0 - \alpha \log k_t < \log s_t < A_2 - c_0 - \alpha \log k_t] = \int_{A_1 - c_0 - \alpha \log k_t}^{A_2 - c_0 - \alpha \log k_t} \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{(x-0)^2}{2\sigma^2}] dx$
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Exercise 2

(i) $\int P(x, A) d\mu(x)$ - is a probability of getting to A tomorrow if we are in $\mu(x)$ today
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(ii) $\int P^n(x, A) \mu_0(dx)$ - is a probability of getting to A in n periods if we were in $\mu_0(x)$ today
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(iii) $\int f(x') P(x, dx')$ - is the expectation of $f(x)$ tomorrow conditional on being in x today
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(iv) $\int f(x') P^n(x, dx')$ - is the expectation of $f(x)$ in n periods conditional on being in x today
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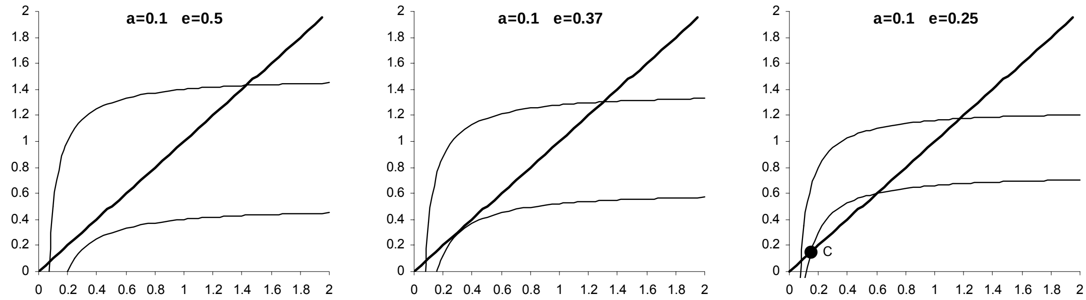
Exercise 3

$x_{t+1} = g(x_t, s_t) = 1 - \frac{a}{x_t} + s_t \qquad S = [-e, e] \nsubseteq R_+$ if $0 < e < 1$

(a)



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(b) Equation $x = 1 - \frac{a}{x} + s$ has 2 solutions if $D = -4a + 2s + s^2 + 1 > 0$

For two solutions for any $s \in [-e, e]$ we need that both $4a < 2e + e^2 + 1$ and $4a < -2e + e^2 + 1$

The resulting condition is: $a < \min[\left(\frac{e+1}{2}\right)^2, \left(\frac{e-1}{2}\right)^2]. = \left(\frac{e-1}{2}\right)^2$

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(c) The point, that we are interested in is marked in the third graph. If we are to the left we could move to the left depending on the realization of the shock. However, if we are to the right of C we remain to the right forever. The value of C is equal to maximum of the two smaller roots:

$x^2 = (1 + s)x - a$, Solution is: $\frac{1}{2}(s + 1) \pm \frac{1}{2}\sqrt{(s + 1)^2 - 4a}$

Hence $C = \max\{\frac{1}{2}(1+e) - \frac{1}{2}\sqrt{(1 + e)^2 - 4a}, \frac{1}{2}(1-e) - \frac{1}{2}\sqrt{(1 - e)^2 - 4a}\} = \frac{1}{2}\left[1 - e - \sqrt{(1 - e)^2 - 4a}\right]$

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(d) $Q(x, [\alpha, \beta]) = \Pr[\alpha \leq x_{t+1} \leq \beta | \, x_t = x] = \Pr[\alpha \leq 1 - \frac{a}{x} + s_t \leq \beta] = \Pr[\alpha + \frac{a}{x} - 1 \leq s_t \leq \beta + \frac{a}{x} - 1] = \Pr[\gamma \leq s_t \leq \delta]$

Though it does not follow from the problem, I guess that the author wanted to say that the distribution of s_t is uniform over $[-e, e]$.

Because from $\alpha < \beta$ if follows that $\gamma < \delta$. Besides $\beta - \alpha = \delta - \gamma$ for any x.

Hence $Q(x, [\alpha, \beta]) = [1 - [\frac{\gamma - (-e)}{2e}]_+ - [\frac{e - \delta}{2e}]_+]_+ = [1 - [\frac{(\alpha + \frac{a}{x} - 1) - (-e)}{2e}]_+ - [\frac{e - (\beta + \frac{a}{x} - 1)}{2e}]_+]_+$

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(e) To prove the existence and uniqueness of a fixed point (limiting probability measure) we need to check condition M. If it holds the defined Markov Process converges to a unique limiting distribution by contraction mapping theorem.

Covering intervals $[\alpha, \beta]$ we cover the whole Borel sigma-algebra because it is the smallest containing all intervals. The limiting distribution is also defined on an interval between the two maximum roots. I.e. if we check it for some $[\alpha, \beta]$ we automatically check it for any $A \in B$.

It's easy to see that the limiting distribution is strictly positive on $[\frac{1}{2}\left(1 - e + \sqrt{(1 - e)^2 - 4a}\right), \frac{1}{2}\left(1 + e + \sqrt{(1 + e)^2 - 4a}\right)]$ and zero everywhere else. It follows from the fact that once we are between the roots we cannot get away and on the other hand it is possible to get from any small interval inside it to any other with nonnegative probability.

Hence there exists some finite N big enough, such that for any ε (and hence there exists one) it then follows that for any interval $[\alpha, \beta]$ it is true that $\Pr[\gamma \leq s_t \leq \delta] \geq \varepsilon$ for any x, or $\Pr[s_t < \gamma \cup \delta < s_t] \geq \varepsilon$ for any x. Hence condition M holds. Q.E.D.