Economics 202A, Homework #2 Anton Cheremukhin

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$\max U = E_1 \left\{ \sum_{t=1}^{\infty} \beta^{t-1} u(C_t, L_t) \right\}, \qquad 0 < \beta < 1$

where $u(C, L) = \frac{\left(C^{\varphi}(1-L)^{1-\varphi}\right)^{1-\rho} - 1}{1-\rho}, \qquad \rho > 0, \qquad 0 < \varphi < 1$

s.t. $K_{t+1} = K_t(1-\delta) + Y_t - C_t, \qquad t = \overline{1\infty}, \qquad K_1 = \bar{K}$

$A_t = A_{t-1}^{\lambda} \exp(e_t), \qquad t = \overline{1\infty}, \qquad 0 < \lambda < 1, \qquad A_0 = \bar{A}$

$Y_t = A_t K_t^{\alpha} \left[(1+g)^t L_t\right]^{1-\alpha}, \qquad 0 < \alpha < 1$

(a) $\lim_{\rho->1} u(x, \rho) = \lim_{\rho->1} \frac{(x)^{1-\rho} - 1}{1-\rho} = \lim_{\rho->1} \frac{(x)^{1-\rho}(-\ln x)}{-1} = \ln x$

(b) $J = E_1 \left\{ \sum_{t=1}^{\infty} \beta^{t-1} \left((K_t(1-\delta) - K_{t+1} + A_t K_t^{\alpha} \left[(1+g)^t L_t\right]^{1-\alpha})^{\varphi} (1-L_t)^{1-\varphi} \right)^{1-\rho} \right\} / (1-\rho)$

FOC: $\frac{\partial J}{\partial K_{t+1}} = \frac{1}{(1-\rho)} E_1 \{ \beta^t ((1-\delta) + \alpha A_{t+1} K_{t+1}^{\alpha-1} \left[(1+g)^{t+1} L_{t+1}\right]^{1-\alpha}) \varphi(1-\rho) \times$

$((1-\delta)K_{t+1} - K_{t+2} + A_{t+1} K_{t+1}^{\alpha} \left[(1+g)^{t+1} L_{t+1}\right]^{1-\alpha})^{\varphi(1-\rho)-1} (1-L_{t+1})^{(1-\varphi)(1-\rho)} -$

$\beta^{t-1} \varphi(1-\rho)((1-\delta)K_t - K_{t+1} + A_t K_t^{\alpha} \left[(1+g)^t L_t\right]^{1-\alpha})^{\varphi(1-\rho)-1} (1-L_t)^{(1-\varphi)(1-\rho)} \} =$

$\varphi \beta^{t-1} \times E_1 [((1-\delta) + \alpha \frac{Y_{t+1}}{K_{t+1}}) \beta \frac{u_{t+1}}{C_{t+1}} - \frac{u_t}{C_t}] = 0$ Euler equation

$\frac{\partial J}{\partial L_t} = \frac{1}{(1-\rho)} \frac{\partial}{\partial L_t} \left((K_t(1-\delta) - K_{t+1} + A_t K_t^{\alpha} \left[(1+g)^t L_t\right]^{1-\alpha})^{\varphi} (1-L_t)^{1-\varphi} \right)^{1-\rho} =$

$[(1-\alpha) \frac{Y_t}{L_t} \varphi \frac{u_t}{C_t} - (1-\varphi) \frac{u_t}{1-L_t}] = 0$ Leasure-consumption trade-off

(c) $k_t = \frac{K_t}{(1+g)^t}, \qquad y_t = \frac{Y_t}{(1+g)^t}, \qquad c_t = \frac{C_t}{(1+g)^t}$

$(1+g)k_{t+1} = k_t(1-\delta) + y_t - c_t \qquad y_t = A_t k_t^{\alpha} L_t^{1-\alpha} \qquad A_t = A_{t-1}^{\lambda} \exp(e_t)$

$E_1 [((1-\delta) + \alpha \frac{y_{t+1}}{k_{t+1}}) \frac{\beta}{1+g} \frac{u_{t+1}}{c_{t+1}} - \frac{u_t}{c_t}] = 0,$

$(1-\alpha) \frac{y_t}{L_t} \varphi \frac{u_t}{c_t} - (1-\varphi) \frac{u_t}{1-L_t} = 0$

$u_t = (C_t^{\varphi}(1-L_t)^{1-\varphi})^{1-\rho} = (c_t^{\varphi}(1-L_t)^{1-\varphi})^{1-\rho} \times (1+g)^{t\varphi(1-\rho)}$

(d) Using the last 3 equations get rid of u_t and L_t :

$(1-\varphi) \frac{c_t}{1-L_t} / ((1-\alpha) \frac{y_t}{L_t} \varphi) = \frac{1-\varphi}{\varphi(1-\alpha)} \frac{L_t}{1-L_t} \frac{c_t}{y_t} = 1, \qquad$ hence $\quad 1 - L_t = \frac{1}{\frac{y_t}{c_t} \frac{\varphi(1-\alpha)}{(1-\varphi)} + 1}$

$\frac{u_t}{c_t} = c_t^{\varphi(1-\rho)-1} [\frac{y_t}{c_t} \frac{\varphi(1-\alpha)}{(1-\varphi)} + 1]^{(\varphi-1)(1-\rho)} \times (1+g)^{t\varphi(1-\rho)}$

$0 = E_1 \{ \beta \frac{(1-\delta) + \alpha \frac{y_{t+1}}{k_{t+1}}}{(1+g)^{1-\varphi(1-\rho)}} c_{t+1}^{\varphi(1-\rho)-1} [\frac{y_{t+1}}{c_{t+1}} \frac{\varphi(1-\alpha)}{(1-\varphi)} + 1]^{(\varphi-1)(1-\rho)} - c_t^{\varphi(1-\rho)-1} [\frac{y_t}{c_t} \frac{\varphi(1-\alpha)}{(1-\varphi)} + 1]^{(\varphi-1)(1-\rho)} \}$

Now assume that no stochastic disturbances occur and we are at steady-state.

$A = A^{\lambda} = 1 \qquad c = y - (\delta + g)k$

$\beta[(1-\delta) + \alpha \frac{y}{k}] = (1+g)^{1-\varphi(1-\rho)}, \qquad \frac{y}{k} = \frac{1}{\alpha} [\frac{1}{\beta}(1+g)^{1-\varphi(1-\rho)} - (1-\delta)] = \theta$

$c/y = 1 - (\delta+g)k/y = 1 - \frac{\delta+g}{\theta} = \varkappa$

$(\frac{y}{c} \frac{\varphi(1-\alpha)}{(1-\varphi)} + 1)(1-L) = 1 \qquad L = 1 - (\frac{1}{\varkappa} \frac{\varphi(1-\alpha)}{(1-\varphi)} + 1)^{-1}$

$y/k = (L/k)^{1-\alpha} \qquad k = L/\theta^{\frac{1}{1-\alpha}}, \qquad y = \theta k, \qquad c = \varkappa y$

(e)

$k_t(1-\delta)+y_t-c_t=(1+g)k_{t+1} \qquad y_t=A_t k_t^{\alpha}L_t^{1-\alpha} \qquad A_t=A_{t-1}^{\lambda}\exp(e_t) \qquad \frac{1-\varphi}{\varphi(1-\alpha)}\frac{L_t}{1-L_t}\frac{c_t}{y_t}=1$

$E_1\{\beta\frac{(1-\delta)+\alpha\frac{y_{t+1}}{k_{t+1}}}{(1+g)^{1-\varphi(1-\rho)}}c_{t+1}^{\varphi(1-\rho)-1}[1-L_{t+1}]^{(1-\varphi)(1-\rho)}-c_t^{\varphi(1-\rho)-1}[1-L_t]^{(1-\varphi)(1-\rho)}\}=0$

Linearize with respect to $\{k_t, y_t, c_t, L_t, A_t\}$ using the following rule:

$0=df(x,y)=x_0\frac{\partial f}{\partial x}|_{x_0,y_0}\frac{x-x_0}{x_0}+y_0\frac{\partial f}{\partial y_0}|_{x_0,y_0}\frac{y-y_0}{y_0}=x_0\frac{\partial f}{\partial x}|_{x_0,y_0}\ln\frac{x}{x_0}+y_0\frac{\partial f}{\partial y_0}|_{x_0,y_0}\ln\frac{y}{y_0}$

$\overline{k}(1-\delta)\ln\frac{k_t}{\overline{k}}+\overline{y}\ln\frac{y_t}{\overline{y}}-\overline{c}\ln\frac{c_t}{\overline{c}}=(1+g)\overline{k}\ln\frac{k_{t+1}}{\overline{k}}$

$\ln\frac{y_t}{\overline{y}}=\ln\frac{A_t}{\overline{A}}+\alpha\ln\frac{k_t}{\overline{k}}+(1-\alpha)\ln\frac{L_t}{\overline{L}}$

$\ln\frac{A_t}{\overline{A}}=\lambda\ln\frac{A_{t-1}}{\overline{A}}+e_t \qquad \ln\frac{y_t}{\overline{y}}=\ln\frac{c_t}{\overline{c}}+\frac{1}{1-\overline{L}}\ln\frac{L_t}{\overline{L}}$

$E_t\{\frac{1}{\frac{1-\delta}{\alpha}\frac{\overline{k}}{\overline{y}}+1}[\ln\frac{y_{t+1}}{\overline{y}}-\ln\frac{k_{t+1}}{\overline{k}}]+[\varphi(1-\rho)-1]\ln\frac{c_{t+1}}{\overline{c}}-(1-\varphi)(1-\rho)\frac{L}{1-L}\ln\frac{L_{t+1}}{\overline{L}}\}=$

$[\varphi(1-\rho)-1]\ln\frac{c_t}{\overline{c}}-\frac{L}{1-L}(1-\varphi)(1-\rho)\ln\frac{L_t}{\overline{L}}+v_{t+1}$

Define more parameters $\phi=\frac{1}{\frac{1-\delta}{\alpha}\frac{\overline{k}}{\overline{y}}+1},\omega=\varphi(1-\rho)-1,\pi=(1-\varphi)(1-\rho)\frac{\overline{L}}{1-\overline{L}}$

Redefine variables

$\phi[E_t\hat{y}_{t+1}-\hat{k}_{t+1}]+\omega[E_t\hat{c}_{t+1}-\hat{c}_t]-\pi[E_t\hat{L}_{t+1}-\hat{L}_t]\}=0$

$-\overline{y}\hat{y}_t+(1+g)\overline{k}\hat{k}_{t+1}+\overline{c}\hat{c}_t=\overline{k}(1-\delta)\hat{k}_t$

$\hat{y}_t-(1-\alpha)\hat{L}_t-\hat{a}_t=\alpha\hat{k}_t$

$\hat{y}_t-\hat{c}_t-\frac{1}{1-\overline{L}}\hat{L}_t=0$

$\hat{a}_t=\lambda\hat{a}_{t-1}+e_t$

$$\begin{bmatrix} 0 & -\phi & -\omega & \pi & 0 & \phi & \omega & -\pi \\ -\overline{y} & (1+g)\overline{k} & \overline{c} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -(1-\alpha) & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & \frac{-1}{1-\overline{L}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_t \\ k_{t+1} \\ c_t \\ L_t \\ a_t \\ E_t y_{t+1} \\ E_t c_{t+1} \\ E_t L_{t+1} \end{bmatrix}=$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{k}(1-\delta) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{t-1} \\ k_t \\ c_{t-1} \\ L_{t-1} \\ a_{t-1} \\ E_{t-1}y_t \\ E_{t-1}c_t \\ E_{t-1}L_t \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}e_t+\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_t \\ w_t \\ u_t \end{bmatrix}$$

(f)
put in values: $\beta = 0.97, g = 0.018, \rho = 2, \alpha = 1/3, \gamma = 1/2, \varphi = 1/2, \lambda = 0.95, \delta = 0.1, \sigma = 0.006$

$$\theta = .4767 \qquad \overline{k} = 1.4275, \qquad \overline{y} = 0.6804, \qquad \overline{c} = 0.512, \qquad \overline{L} = 0.4698$$

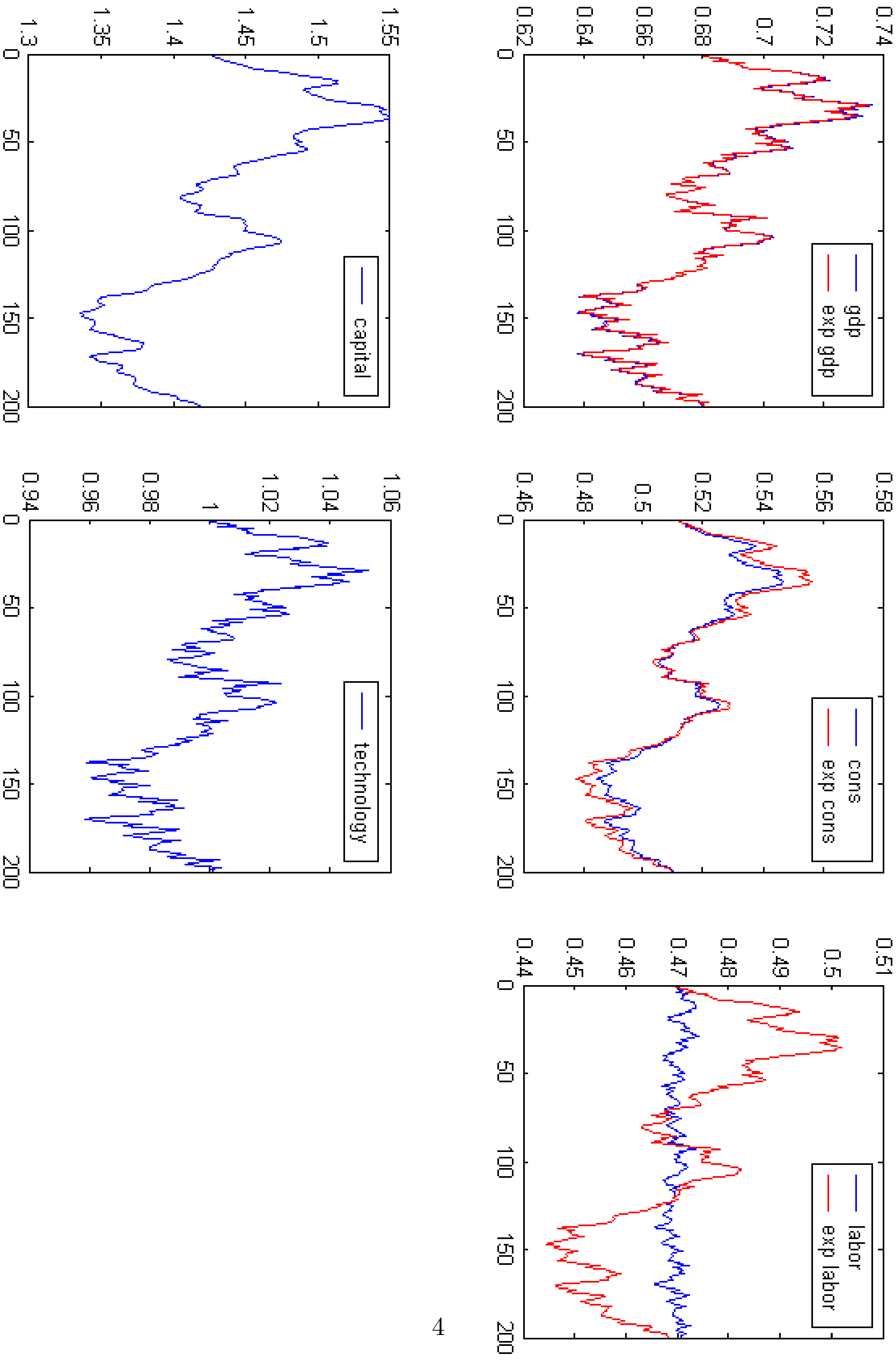
$$\phi = .1501, \qquad \omega = -1.5, \qquad \pi = -.443$$

$$\begin{bmatrix} 0 & -.15 & 1.5 & -0.443 & 0 & .15 & -1.5 & .443 \\ -0.68 & 1.4532 & .512 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2/3 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1.886 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ k_{t+1} \\ c_t \\ L_t \\ a_t \\ E_t y_{t+1} \\ E_t c_{t+1} \\ E_t L_{t+1} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.2847 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .95 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ k_t \\ c_{t-1} \\ L_{t-1} \\ a_{t-1} \\ E_{t-1} y_t \\ E_{t-1} c_t \\ E_{t-1} L_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e_t + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_t \\ w_t \\ u_t \end{bmatrix}$$

Gensys gives the following solution:

$$\begin{bmatrix} y_t \\ k_{t+1} \\ c_t \\ L_t \\ a_t \\ E_t y_{t+1} \\ E_t c_{t+1} \\ E_t L_{t+1} \end{bmatrix} = \begin{bmatrix} -0.2248 & 1.1969 \\ 0.802 & 0.3848 \\ 0.5318 & 0.4985 \\ -0.1628 & 0.3703 \\ 0 & 0.95 \\ 0.1803 & 1.2235 \\ 0.6431 & 0.6741 \\ 0.4265 & 0.6782 \end{bmatrix} \begin{bmatrix} k_t \\ a_{t-1} \end{bmatrix} + \begin{bmatrix} 1.2598 \\ 0.4050 \\ 0.5247 \\ 0.3898 \\ 1 \\ 0.2879 \\ 0.7096 \\ 0.7139 \end{bmatrix} e_t$$


gdp
exp gdp
cons
exp cons
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exp labor
capital
technology