Economics 201C
Spring 2006
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Homework 4

1. Price-taking equilibrium and efficiency

There are two types of individuals, A and B and there is a large but *equal* number of them, say 1000. Type A can choose to be in only one of three possible occupations. His utility in each is

$$\begin{aligned}
v_A^{\#1}(z_1,z_2) &= \min\{z_1+1, z_2+1\} \text{ if } z_1, z_2 \geq -1, \quad -\infty \text{ otherwise} \\
v_A^{\#2}(z_1,z_2) &= \min\{z_1+3, z_2\} \text{ if } z_1 \geq -3, z_2 \geq 0, \quad -\infty \text{ otherwise} \\
v_A^{\#3}(z_1,z_2) &= \min\{z_1+4, z_2\} \text{ if } z_1 \geq -4, z_2 \geq 0, \quad -\infty \text{ otherwise}
\end{aligned}$$

In other words, A can choose to have an endowments of either $(1,1)$, $(3,0)$, or $(4,0)$ while having the same utility function. Type B individuals have the same utilities, but their occupations are the symmetric opposite of type A, with endowments either $(1,1)$, $(0,3)$, or $(0,4)$.

(a) Suppose both types choose occupations #3. Find the price-taking equilibria. (Suggestion: Can you ignore trade in the money commodity? Are quantities unique? Are prices unique?)

(b) Show that for any equilibrium price vector (p_1,p_2) in (a), there is always at least one individual who, by switching to his second occupation, could improve his utility in a price-taking equilibrium.

(c) Suppose both types were to choose occupation #1. Show that if any individual were to change to occupation #2 or #3, his utility would not improve.

(d) The conclusion from (b) and (c) is that choosing occupation #3 is not an equilibrium and choosing occupation #1 is. Since #3 is the only choice compatible with efficiency, how can you reconcile your conclusion with the First Theorem of Welfare Economics?

2. Externalities in Games

Suppose $v_1(z_1,z_2)$ and $v_2(z_2,z_1)$ are the payoff functions in a normal form game in which each $z_i \in [0,\bar{x}]$ so there is a continuum of strategies. Assume each v_i is concave and differentiable.

(a) Define a Pareto-efficient play of the game. Explain why such a play can be characterized as maximizing the weighted sum $\lambda_1 v_1(z_1,z_2) + \lambda_2 v_2(z_1,z_2)$ among all possible plays of the game, where λ_i are non-negative and sum to 1. Why is concavity of v_i important? Do you need to assume quasilinearity to justify your answer?

(b) What are the FOC for a Pareto efficient play of the game [call it $z^o = (z_1^o, z_2^o)$]? If a play of the game satisfies the FOC for efficiency, why does that imply it is efficient?

(c) What are the first order conditions for a Nash equilibrium play of the game [call it $z^e = (z_1^e, z_2^e)$]? Explain why λ_i weights play no role. If a play of the game satisfies the FOC for Nash equilibrium, why does that imply it is a Nash equilibrium?

Suppose

$$\begin{aligned} v_1(z_1, z_2) &= A_{11}z_1 - B_{11}z_1^2/2 + A_{12}z_2 - B_{12}z_2^2/2 \\ v_2(z_2, z_1) &= A_{22}z_2 - B_{22}z_2^2/2 + A_{21}z_1 - B_{21}z_1^2/2 \end{aligned}$$

where the constants A_{ij} and B_{ij} are positive.

(c) Find z^o and z^e as functions of the constants. [Assume $\bar{x}$ is sufficiently large that it is not a binding constraint. Also assume that the weights on each person used to calculate z^o are the same and equal to 1.]

(d) Find conditions such that (i) z_1 is undersupplied in equilibrium [$z_1^e < z_1^0$], (ii) z_1 is oversupplied in equilibrium [$z_1^e > z_1^0$] and (iii) z_1 is efficiently supplied in equilibrium [$z_1^e = z_1^o$].

(e) Use the Pigovian heuristic comparing private and social benefits and costs to explain undersupply and oversupply in (d). How can you explain (iii)?

3. The Geometry of the Coordination Problem

The directional derivative of f at $z = (z_1, z_2)$ in the direction $d = (d_1, d_2)$ is:

$$Df(z; d) = \lim_{t \searrow 0} \frac{f(z + td) - f(z)}{t}$$

Two facts about the directional derivative are:

- If f is concave, but not necessarily differentiable, the directional derivative is super-additive: $Df(z; d + d') \geq Df(z; d) + Df(z; d')$
- If f is differentiable, but not necessarily concave, the directional derivative is linear: there is a $p = (p_1, p_2)$ such that $pd = Df(z; d)$ for all d. Hence, $p(d + d') = Df(z; d) + Df(z; d) = Df(z; d + d')$.

Problem 2. assumed that each v_i was concave and differentiable. Therefore so is the weighted sum function $v(z) = [\lambda_1 v_1 + \lambda_2 v_2](z)$. Throughout the following, assume the weights are equal. Also, for parts (a)–(d), assume v_i is concave but not necessarily differentiable.

(a) For a feasible z, what are the necessary and sufficient conditions on $Dv(z;d)$ for it to be Pareto efficient? [Assume $z \gg 0$.]

(b) Explain why the conditions $Dv_1(z;(1,0))$, $Dv_1(z;(-1,0))$, $Dv_2(z;(0,1))$ and $Dv_1(z;(0,-1))$ all ≤ 0 are necessary and sufficient conditions for z to be a Nash equilibrium.

(c) Suppose (I) $Dv_1(z;(\alpha,0)) = Dv(z;(\alpha,0))$ and $Dv_2(z;(0,\alpha)) = Dv(z;(0,\alpha))$ for all α, positive or negative. Suppose, in addition, that (II) z satisfies the conditions in (b) for Nash equilibrium. Explain why the combination of (I) and (II) is the marginalist version of the Pigovian recipe for the elimination of the harmful effects of externalities.

(d) Explain/illustrate why the condition in (c) does NOT suffice for efficiency? Explain why condition (c) would suffice if there were differentiability?

Suppose

$$
\begin{aligned}
v_1(z_1,z_2) &= z_2(A_1 z_1 - B_1 z_1^2/2) - z_2^{1/2}/2 \\
v_2(z_1,z_2) &= z_1(A_2 z_2 - B_2 z_2^2/2) - z_1^{1/2}/2
\end{aligned}
$$

(e) Verify that v_i are not concave. (They are of course differentiable.)

(f) Let $A_i = B_i = 1$. Show that there is a Nash equilibrium at $z_1 = z_2 = 1$.

(g) Show that at this Nash equilibrium $\frac{\partial v(z)}{\partial z_i} = 0$. Therefore, it satisfies the differentiable version of (c).

(h) Show that the failure of concavity is consistent with the possibility that there are coordination problems. Find $(\tilde{z}_1, \tilde{z}_2)$ that are Pareto improvements on $z_1 = z_2 = 1$.