Economics 201C
Spring 2006
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Homework 2 (LP and QL)

1. The Value Function

Activity 1 is the cutting of trees on one acre of forest land and activity 2 is cutting and replanting per acre. $A^1 = (1, \$10)$ and $A^2 = (1, \$50)$ indicating that it takes one acre and certain amounts of money to operate each activity at unit level. The revenues are $c_1 = 40$ and $c_2 = 70$. The resource constraint is $b = (100, \$4000)$, indicating the number of acres available and the funds available to pay for the activities. (Revenues are received later and cannot be borrowed against.)

[Illustrate your answers to (b)–(f).]

(a) Find the optimal solutions to the primal and dual.

(b) Suppose the individual could borrow \$t dollars, i.e., make the second constraint $\$4000+t$. Use the information in (a) to give an *estimate* of the maximum interest rate r that should be paid per dollar. Similarly, suppose the individual could lend \$t dollars, i.e., make the constraint $\$4000-t$. Give an estimate of the minimum return s the individual would accept to lend.

(c) Show that there is a range for which the estimate in (b) is actually correct both for borrowing and for lending. Find the range $[\$4000 - t_1, \$4000 + t_2]$ such that the estimate in (b) is correct.

(d) How does the estimate compare with the true answer beyond $\$4000 + t_2$ and how does it compare below $t_1 - \$4000$?

(e) Explain why the questions above involve comparisons between $y(a-b)$, where $y \in \partial h(b)$ and $h(a) - h(b)$ where $a = (100, a_2)$ for all a_2.

(f) Suppose that instead of $b = (100, \$4000)$, it was $(100, \$5000)$. How would your answer to the question in (c) change with respect to the range $[\$5000 - t_1, \$5000 + t_2]$? What is the dual solution when $b = (100, 5001)$?

2. Pricing of Tradeable Goods and Non-Tradeable Constraints

There are three firms using inputs and producing outputs of three commodities. Each firm j has only one activity $A^j = (a_{1j}, a_{2j}, a_{3j})$, where a_{ij} is the amount of commodity i needed to operate activity (firm) j at unit level. (Recall that $a_{ij} < 0$ means that i is an output of j.) Let $A^1 = (-5, 2, 1)$, $A^2 = (2, -3, 1)$, and $A^3 = (0, 2, -3)$. The unit revenues associated with each firm-activity are $c = (c_1, c_2, c_3) = (2, 0, 1)$. (The unit revenue from an activity is not the same as value of the negative a_{ij}. Thus, A^2 is a purely intermediary activity having no direct market value.)

Each firm-activity operates under a capacity constraint that its activity level cannot be greater than 1. Finally, existing stocks of the three commodities are $s = (s_1, s_2, s_3) = (0, 0, 0)$, i.e., all outputs must be produced from inputs.

(a) Write the linear programming problem for maximizing total revenue subject to the constraints, and then write out the dual.

(b) Verify that $\overline{x} = (\overline{x_1}, \overline{x_2}, \overline{x_3}) = (7/8, 1, 5/8)$ is an optimal solution to the primal. (Suggestion: Is it feasible? Use Complementary Slackness to find a feasible solution to the dual; and then use the Optimality Criterion to show that both this feasible solution for the dual and the given solution for the primal are optimal.)

(c) Assuming that capacity constraints are not priced but the three commodities used as inputs and outputs are, use the answers in (b) to find a price equilibrium to decentralize decision-making among the three firms.

(d) How do the resulting profits from the decentralized pricing solution to the revenue maximization problem relate to the dual solutions on the capacity constraints? Explain.

3. Demand Theory with Quasi-linear Utility

In the usual formulation of the consumer's problem (not the net trade version), if a consumer's utility function is $u(x, m) = v(x) + m$, where $x \in \mathbf{R}^\ell_+$ and $m \in \mathbf{R}_+$ and the budget constraint is $px + m = w$, show that

(a) the indirect utility function is of the form $\alpha(p) + w$

(b) the expenditure function is of the form $\beta(p) + u$

(c) the demand for the first ℓ commodities is *independent* of the level of income

(d) own price elasticities are always negative.

4. Demand/Supply, Inverse Demand/Supply and Indirect Utility/Profit

(a) Show that the following conditions on p and z are equivalent:

(1) (Demand) $(z, m) \in D(p)$, where $D(p) = \text{argmax}\{v(z) + m : pz + m = 0\}$

(2) (Inverse demand) $p \in \partial v(z)$

(3) (Indirect Utility) $v^*(p) = v(z) - p \cdot z$

(b) Suppose $v = -c(z)$, where $c(z)$ is the money cost (positive (possibly $+\infty$), negative, or zero) associated with production of the vector of non-money commodities. Interpret (1)–(3) for a producer. I.e., what are the analogs of (1)–(3) for a producer and what are the corresponding equivalent conditions on p and z.