Economics 201C
Spring 2006
Instructor: J. Ostroy

Homework 1 (Standard Model)

1. Replica Invariance and Convexity

There are two commodities, x and y. Each individual has an endowment $(1,1)$. Preferences are represented by utility functions of the form

$$u_\alpha = \alpha x^2 + y^2 \qquad \text{or} \qquad v_\alpha = \alpha x^{1/2} + y^{1/2}.$$

Prices are $p = (1,b)$, i.e., b is the ratio of the price of y to the price of x. Therefore, individual wealth is $w_i(p) = (1+b)$.

(a) Find utility maximizing demand $D(b)$ as a function (or correspondence) of b for an individual with utility u_α.

(b) There are two individuals with preferences distinguished by α_I and α_{II}. Show that there is a price-taking equilibrium for any $\alpha_I, \alpha_{II} > 0$. E.g., $\alpha_I = 2, \alpha_{II} = 1/2$.

(c) Confirm the replica invariance property of price-taking equilibrium: the equilibrium in (b) implies an equilibrium for any economy consisting of k each of the two types of individuals.

(d) (A) Show that there there does NOT exist a price-taking equilibrium when there are two individuals of type I and one of type II. E.g., when $\alpha_I = 2, \alpha_{II} = 1/2$. (B) Show that the same NON-existence result applies when $\alpha_I = \alpha_{II} = 1$ so that there are three individuals of the same type. What about when there are k of type I and $k-1$ of type II, or an odd number of individuals all of the same type?

(e) Show that there is a price-taking equilibrium when the model in (d.A) is replicated so that there are 4 of type I and 2 of type II. Note: This example demonstrates that, in general, replica invariance for *scaling up* does not imply replica invariance for *scaling down*.

(f) Find $D(b)$ for v_α and price-taking equilibrium with two individuals, $\alpha_I, \alpha_{II} > 0$.

(g) Show that for any k, odd or even, there is a price-taking equilibrium when all individuals have the same utilities v_α. Find the equation for the equilibrium value of b and the equilibrium allocation. [Note: It is enough to find the equation for market clearance for the second commodity (or first). Why?]

(h) To demonstrate that there are no existence problems when preferences can be represented by a function of the form v_α, suppose there are k individuals with utilities v_{α_i}, $i = 1, \ldots, k$, and α_i are possibly all distinct. Find the equation for the equilibrium value of b.

(h) When there is an even number of individuals, price-taking exists when all individuals have utilities u_α or v_α. How do they differ? Why?

2. Replica Invariance in Games/Economies

(a) A key reason why replica invariance can be defined for economic models — whether or not it holds — is: the characteristics of any one (consumer or producer) are described independently of the characteristics or the behavior of any other. Explain why the property of replica invariance is not meaningful, or at least difficult to define, for games. E.g., how would you define the "replica" of a 2×2 game? How many strategies does each individual have? What are individual payoffs?

(b) Suppose a two person model of exchange in which i's utility also depends on j's consumption, i.e., $u_i(x_i, x_j)$, $i, j = 1, 2$, $i \neq j$. What difficulties would be encountered in defining replica invariance? E.g., how would you define the four-person model?

3. Wealth Distribution and Efficiency

Using the notation in Debreu's *Theory of Value* and the class notes, define an "equilibrium relative to a wealth system" as an allocation $[(x_i^*), (y_j^*), p]$ for the economy

$$E^\alpha = \{X_i, (\succeq_i), (Y_j), \omega\},$$

such that

- $x_i^* \in \xi_i(p, w_i(p))$ for all i,
- $y_j^* \in \eta_j(p)$ for all j,
- $\sum_i x_i^* - \sum_j y_j^* = \omega$

The wealth system for E^α is

$$w_i(p) = \alpha_i(p) w(p),$$

where $w(p) = p\omega + \sum_j \pi_j(p)$, and $\alpha_i(p) > 0$, $\sum \alpha_i(p) = 1$ for all p. Hence i's share of total wealth, $\alpha_i(\cdot)$, may depend on p.

(a) Demonstrate that an equilibrium relative to a wealth system is (weakly) optimal. [Requires a brief but formal argument.] If α_i does not depend on p does your answer change?

(b) Show that by specializing the wealth system, the result in (a) can be used to demonstrate the First Theorem of Welfare Economics. [Requires a *particular* definition of $w_i(p)$.]

(c) (1) What is the Second Theorem of Welfare Economics? (2) What conditions are required for its validity? [Requires a statement in words, possibly including symbols, for (1) and geometric arguments (examples) to support your answer for (2).]