Econ201C Homework #4 Anton Cheremukhin

June 6, 2006

Exercise 1 *Price-taking equilibrium and real externalities*

$v(x,z) = 2\left(x^{1/2} + \alpha z^{1/2}\right) \qquad z = f(y) \qquad c(y) = y^2/2 \qquad f(y) = y \qquad \alpha \geq -1$
Define $v^*(p,\overline{z}) = \max_{x'} \{v(x',\overline{z}) - px' | x' \geq 0\}$
Define $\pi^*(p) = \max_{y'} \{py' - c(y') | y' \geq 0\}$
Price-taking equilibrium: allocation $\{x,y,z\}$ and price p s.t.
a) $v^*(p,\overline{z}) = v(x,\overline{z}) - px, \qquad p \in \partial v(x,\overline{z}), \qquad x \geq 0$
b) $\pi^*(p) = py - c(y), \qquad p \in \partial c(y), \qquad y \geq 0 \qquad$ c) $y = x, \qquad \overline{z} = f(y)$
For our functions: $p = \frac{\partial}{\partial x} 2\left(x^{1/2} + \alpha \overline{z}^{1/2}\right) = x^{-1/2} \qquad p = \frac{\partial}{\partial y}(y^2/2) = y \qquad y = x \qquad \overline{z} = y$
Therefore, $x = y = z = p = 1$. It is independent of α.
Efficient allocation: $\{x,y,z\} = \arg\max \{v(x,z) - c(y) | z = f(y), x = y\} =$
$= \arg\max \{v(y, f(y)) - c(y) | z = f(y), x = y\}$
For our functions: $0 = \frac{\partial}{\partial y}\left(2(1+\alpha) y^{1/2} - y^2/2\right) = \frac{1}{\sqrt{y}}(\alpha+1) - y \qquad \Rightarrow \qquad y = (\alpha+1)^{2/3}$.
If $\alpha > 0$ then too little is consumed in equilibrium, the real externality is positive.
If $\alpha < 0$ then too much is consumed in equilibrium, the real externality is negative.
Marginal private benefit of the consumer is positive while marginal social benefit is negative.
So the incentives are not properly alligned, and the price-taking equilibrium is not efficient.
Pigovian tax: $v^*(p,\overline{z}) = \max_{x'} \{v(x',\overline{z}) - (p+t)x' + t\overline{x} | x' \geq 0\}$
$p + t = \frac{\partial}{\partial x} 2\left(x^{1/2} + \alpha \overline{z}^{1/2}\right) = x^{-1/2} \qquad p = \frac{\partial}{\partial y}(y^2/2) = y = x \qquad \Rightarrow \qquad y + t = \frac{1}{\sqrt{y}}$
Need to replicate $y = (\alpha+1)^{2/3}$. Hence, optimal $t^* = (\alpha+1)^{-1/3} - (\alpha+1)^{2/3}$.
Now define $v^*(p,r) = \max_{x'} \{v(x',z') - px' - rz' | x' \geq 0, z' \geq 0\}$
Define $\pi^*(p,r) = \max_{y'} \{py' + rf(y') - c(y') | y' \geq 0\}$
Complete markets price-taking equilibrium: allocation $\{x,y,z\}$ and price $\{p,r\}$ s.t.
a) $v^*(p,r) = v(x,z) - px - rz, \qquad p \in \partial_x v(x,z), \qquad r \in \partial_z v(x,z), \qquad x \geq 0, \qquad z \geq 0$
b) $\pi^*(p,r) = py + rf(y) - c(y), \qquad p \in \partial(c(y) - rf(y)), \qquad y \geq 0 \qquad$ c) $y = x, \qquad z = f(y)$
For our functions: $p = \frac{\partial}{\partial x} 2\left(x^{1/2} + \alpha z^{1/2}\right) = x^{-1/2} \qquad r = \frac{\partial}{\partial z} 2\left(x^{1/2} + \alpha z^{1/2}\right) = \alpha z^{-1/2}$
$p = \frac{\partial}{\partial y}(y^2/2 - ry) = y - r \qquad y = x \qquad z = y.$
Therefore, $x = y = z = p + r = (\alpha+1)^{2/3}$. Now the price-taking equilibrium is efficient.

Exercise 2 *Resource Allocation for a Public Park*

$x \in [0,1] = X$ - fraction of land devoted to sports fields. Costs are zero, independent of x.
n quasilinear users with $v_i = a_i x - b_i x^2, 0 \leq a_i \leq 2b_i$
The allocation $x = \arg\max \{\Sigma_{i=1}^n v_i(x) | x \in X\}$ of public good is optimal (socially efficient).
FOC: $\qquad \Sigma_{i=1}^n \frac{\partial}{\partial x} v_i(x^*) = 0. \qquad$ Assume that $x^* \in X$.
Lindahl equilibrium is a vector $\{(p_i), p_0, (x_i), x, (m_i)\}$ s.t.
a) $x_i - x \leq 0 \qquad$ b) $\Sigma_{i=1}^n p_i - p_0 = 0 \qquad$ c) $m_i = -p_i x_i$
d) $x_i = \arg\max_{x_i} \{v(x_i) - p_i x_i | x_i \in X\} \qquad$ e) $x = \arg\max_x \{p_0 x | x \in X\}$
FOC: $\frac{\partial}{\partial x_i} v_i(x_i) = p_i \qquad p_0 = 0$
Besides, $\qquad x_i = x \qquad 0 = p_0 = \Sigma_{i=1}^n p_i = \Sigma_{i=1}^n \frac{\partial}{\partial x_i} v(x_i) = \Sigma_{i=1}^n \frac{\partial}{\partial x} v_i(x)$.
Therefore, Lindahl equilibrium is efficient.

Define $g(\mathbf{v}) = \max\{\Sigma_{i=1}^n v_i(x_i) \,|x_i = x \in X\}$ $\qquad MP_i(\mathbf{v}|v_i) = g(\mathbf{v}) - g(\mathbf{v}_{-i})$

Let the **mechanism** be: $x(\mathbf{v}|v_i) = \arg\max g(\mathbf{v}|v_i)$, $\quad m_i(\mathbf{v}|v_i) = g(\mathbf{v}|v_i) - g(\mathbf{v}_{-i}) - v_i(x(\mathbf{v}|v_i))$

This mechanism gives everybody their marginal product:

$\pi_i^F(\mathbf{v}|v_i) = v_i(x(\mathbf{v}|v_i)) + m_i(\mathbf{v}|v_i) = g(\mathbf{v}|v_i) - g(\mathbf{v}_{-i}) = MP_i(\mathbf{v}|v_i)$:

This mechanism is also:

a) efficient in the non-money commodity under true reports: $x(\mathbf{v}) \in \arg\max g(\mathbf{v})$

b) not manipulable by i: $\pi_i^F(\mathbf{v}) = \max_{v_i'} \pi_i^F(\mathbf{v}|v_i'; v_i)$, since:

If the guy decides to misrepresent: $x(\mathbf{v}|v_i', v_i)$, $m_i(\mathbf{v}|v_i', v_i)$, $v_i' \neq v_i$ - then

$\pi_i^F(\mathbf{v}|v_i') = g(\mathbf{v}|v_i') - g(\mathbf{v}_{-i}) < g(\mathbf{v}|v_i) - g(\mathbf{v}_{-i}) = \pi_i^F(\mathbf{v}|v_i)$ - his decision is suboptimal.

Money balance:

$\Delta_i(\mathbf{v}) = m_i(\mathbf{v}) - m_i^L(\mathbf{v}) = g(\mathbf{v}) - g(\mathbf{v}_{-i}) - v_i(x(\mathbf{v})) + x(\mathbf{v}) \frac{\partial}{\partial x} v_i(x(\mathbf{v}))$

$\Sigma_{i=1}^n \Delta_i(\mathbf{v}) = \Sigma_{i=1}^n \left(MP_i(\mathbf{v}) - v_i(x(\mathbf{v})) + x(\mathbf{v}) \frac{\partial}{\partial x} v_i(x(\mathbf{v}))\right) = \Sigma_{i=1}^n MP_i(\mathbf{v}) - g(\mathbf{v}) \leq 0$,

since $\Sigma_{i=1}^n MP_i(\mathbf{v}) \leq g(\mathbf{v})$ for the public good model.

Proof: $MP_i(\mathbf{v}) = g(\mathbf{v}) - g(\mathbf{v}_{-i}) = g(\mathbf{v}) - \max\{\Sigma_{j\neq i} v_j(x_j) \,|x_j = x \neq x_i\}_U \leq$

$\leq g(\mathbf{v}) - \max\{\Sigma_{j\neq i} v_j(x_j) \,|x_j = x_i\}_R = v_i(x^*)$ $\qquad \Rightarrow \qquad \Sigma_{i=1}^n MP_i(\mathbf{v}) \leq \Sigma_{i=1}^n v_i(x^*) = g(\mathbf{v})$

Ex: $v_i = a_i x - b_i x^2$ $\qquad p_i = \frac{\partial}{\partial x} v_i = a_i - 2b_i x$ $\qquad \Sigma_{i=1}^n p_i = \Sigma_{i=1}^n a_i - 2x \Sigma_{i=1}^n b_i = 0$ $\qquad x = \frac{\overline{a}}{2\overline{b}}$

In some special cases it can fulfil as equality. If $a_i/2b_i = \overline{a}/2\overline{b}$ is the same for all agents, then $p_i = 0$, and x does not change if an agent leaves.

Hence, $MP_i(\mathbf{v}) = v_i(x)$, $\Sigma_{i=1}^n MP_i(\mathbf{v}) = g(\mathbf{v})$, $\Sigma_{i=1}^n \Delta_i(\mathbf{v}) = 0$.

This is the case of full appropriation.