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Exercise 1

a) Type I: $u_\alpha = \alpha x^2 + y^2 \to \max_{x,y}$ s.t. $x + by \le 1 + b$

This is not a convex utility function, so we shall always get corner solutions. So,

$x = 1 + b$ if $\alpha x^2 \ge y^2$ $\quad$ $y = 1/b + 1$ if $\alpha x^2 \le y^2$ $\quad$ Switch at: $\alpha (1+b)^2 \ge \left(\frac{1+b}{b}\right)^2$

I.e. Demand $\xi_u(b) = \begin{cases} (1+b, 0), \alpha b^2 \ge 1 \\ (0, 1+1/b), \alpha b^2 \le 1 \end{cases}$

General case: k individuals with α_I and m individuals with α_{II} $\quad$ $(u_\alpha = \alpha_i x^2 + y^2)$

$$k \begin{cases} (1+b, 0), \alpha_I b^2 \ge 1 \\ (0, 1+1/b), \alpha_I b^2 \le 1 \end{cases} + m \begin{cases} (1+b, 0), \alpha_{II} b^2 \ge 1 \\ (0, 1+1/b), \alpha_{II} b^2 \le 1 \end{cases} = (k+m, k+m)$$

$\alpha_I \ge \left(\frac{k}{m}\right)^2, \alpha_{II} \le \left(\frac{k}{m}\right)^2 \quad \Rightarrow \quad k(1+b) = k+m,\ m\left(1+\frac{1}{b}\right) = k+m \quad b = \frac{m}{k}$

$\alpha_I \le \left(\frac{m}{k}\right)^2, \alpha_{II} \ge \left(\frac{m}{k}\right)^2 \quad \Rightarrow \quad k\left(1+\frac{1}{b}\right) = k+m, m(1+b) = k+m \quad b = \frac{k}{m}$

(We here assume that all same types choose the same allocation.

Indifference between corners - is easy to check by hand and is mere coincidence).

For other combinations of (α_I, α_{II}) the equilibrium does not exist.

b) k=m=1 $\quad$ $\alpha_I \ge 1, \alpha_{II} \le 1, b = 1$ $\quad$ or $\quad$ $\alpha_I \le 1, \alpha_{II} \ge 1, b = 1$

For $[\alpha_I < 1, \alpha_{II} < 1]$ or $[\alpha_I > 1, \alpha_{II} > 1]$ the equilibrium does not exist.

E.g. for $\alpha_I = 2, \alpha_{II} = 1/2$, $b = 1$, $x_I = 2$, $x_{II} = 0$, $y_I = 0$, $y_{II} = 2$.

c) k=m $\quad$ Same as (b).

d.A) k=2, m=1 $\quad$ $\alpha_I \ge 4, \alpha_{II} \le 4 \Rightarrow b = 1/2$ $\quad$ $\alpha_I \le 1/4, \alpha_{II} \ge 1/4 \Rightarrow b = 2$

E.g. for $\alpha_I = 2, \alpha_{II} = 1/2$ the equilibrium does not exist.

The indifference case implies $b = 1/\sqrt{2}$. This is not an equilibrium either.

d.B) If there is an odd number of identical guys $\alpha_I = \alpha_{II} = 1$: there is a coordination problem

$k \begin{cases} (1+b, 0), \alpha b^2 \ge 1 \\ (0, 1+1/b), \alpha b^2 \le 1 \end{cases} = k(1,1)$ $\quad$ - no equilibrium even for b=1.

For an even number b=1 is the equilibrium price, because then half can choose the first option and half - the second option.

d.C) $m = k - 1$

$\alpha_I \ge \left(\frac{k}{k-1}\right)^2, \alpha_{II} \le \left(\frac{k}{k-1}\right)^2 \Rightarrow b = \frac{k-1}{k}$ $\quad$ $\alpha_I \le \left(1 - \frac{1}{k}\right)^2, \alpha_{II} \ge \left(1 - \frac{1}{k}\right)^2 \Rightarrow b = \frac{k}{k-1}$

E.g. $\alpha_I = 2, \alpha_{II} = 1/2$ the equilibrium does not exist for k=1,2,3. For k>3 it exists.

For $\alpha_I = \alpha_{II} = 1$ equilibrium does not exist.

e) k=4, m=2 $\quad$ $\alpha_I \ge 4, \alpha_{II} \le 4 \Rightarrow b = 1/2$ $\quad$ $\alpha_I \le \frac{1}{4}, \alpha_{II} \ge \frac{1}{4} \Rightarrow b = 2$

E.g. $\alpha_I = 2, \alpha_{II} = 1/2$ do not fit here. $\quad$ Check indifference case: $b = 1/\sqrt{2}$:

$k(1+b) = 6$ does not support it for any k. $\quad$ Hence, there is no equilibrium.

f) Type II: $v_\alpha = \alpha x^{1/2} + y^{1/2} \to \max_{x,y}$ s.t. $x + by \le 1 + b$

$\alpha x^{1/2} + y^{1/2} + \lambda (1 + b - x - by) \to \max_{x,y,\lambda}$

FOC: $\alpha \frac{1}{2} x^{-1/2} = \lambda$ $\quad$ $\frac{1}{2} y^{-1/2} = \lambda b$ $\quad$ $1 + b = x + by$

$y = \frac{1+b}{(\alpha b)^2 + b}$ $\quad$ $x = (\alpha b)^2 \frac{1+b}{(\alpha b)^2 + b}$ $\quad$ Demand: $\xi_v(b) = \left((\alpha b)^2 \frac{1+b}{(\alpha b)^2+b}, \frac{1+b}{(\alpha b)^2+b}\right)$

PTE: $\left((\alpha_I b)^2 \frac{1+b}{(\alpha_I b)^2+b}, \frac{1+b}{(\alpha_I b)^2+b}\right) + \left((\alpha_{II} b)^2 \frac{1+b}{(\alpha_{II} b)^2+b}, \frac{1+b}{(\alpha_{II} b)^2+b}\right) = (2,2)$

Can use only one equation due to strict concavity of v_α and hence validity of Walras law.
$\left((\alpha_{II}b)^2+(\alpha_I b)^2+2b\right)(1+b)=2\left((\alpha_I b)^2+b\right)\left((\alpha_{II}b)^2+b\right)$
$g(x)=2\alpha_I^2\alpha_{II}^2x^3+\left(\alpha_{II}^2+\alpha_I^2\right)x^2-\left(\alpha_{II}^2+\alpha_I^2\right)x-2=0$
A cubic equation always has at least one solution.
This solution is positive since $g(0)<0$, $g(+\infty)=+\infty$, and g() is continuous.
For $\alpha_I=2,\alpha_{II}=1/2$ the non-negative solution is: $b=1$

g) PTE: $k\left((\alpha b)^2\frac{1+b}{(\alpha b)^2+b},\frac{1+b}{(\alpha b)^2+b}\right)=k(1,1) \qquad b=1/\alpha$
Autarky is the equilibruim allocation.

h) PTE: $\Sigma_{i=1}^k\left((\alpha_i b)^2\frac{1+b}{(\alpha_i b)^2+b},\frac{1+b}{(\alpha_i b)^2+b}\right)=k(1,1)$. Due to strict concavity a solution exists.

i) For an even number of u_α with same $\alpha=1$ the equilibrium price is (1,1). In equilibrium the agents exchange goods: some of them consume only good 1, others - only good 2.
For an even number of u_α with same $\alpha\neq1$ or for an odd number there is no equilibrium.
For any number of v_α with different α_i there is an equilibrium.
For identical v_α-agents with autarky and $b=1/\alpha$ is the equilibrium.
All these differences are due to non-convexity of u_α and convexity of v_α.

j) Mixing types: Let there be 1 individual of type I and 1 individual of type II:

$$\text{PTE: }\left((\alpha_{II}b)^2\frac{1+b}{(\alpha_{II}b)^2+b},\frac{1+b}{(\alpha_{II}b)^2+b}\right)+\begin{cases}(1+b,0),\alpha_I b^2\geq1\\(0,1+1/b),\alpha_I b^2\leq1\end{cases}=(2,2)$$

$\boxed{\alpha_I b^2\geq1}$: $\quad\left[(\alpha_{II}b)^2\frac{1+b}{(\alpha_{II}b)^2+b}+1+b=2,\ \frac{1+b}{(\alpha_{II}b)^2+b}=2\right]\quad\Rightarrow\quad 2(\alpha_{II}b)^2+1+b=2$

$2\alpha_{II}^2b^2+b-1=0 \qquad D=1+8\alpha_{II}^2 \qquad \boxed{b=\frac{\sqrt{1+8\alpha_{II}^2}-1}{4\alpha_{II}^2}}$

$\boxed{\alpha_I b^2\leq1}$: $\quad(\alpha_{II}b)^2\frac{1+b}{(\alpha_{II}b)^2+b}=2,\ \frac{1+b}{(\alpha_{II}b)^2+b}+\frac{1+b}{b}=2\quad\Rightarrow\quad(\alpha_{II}b)^2\left[2-\frac{1+b}{b}\right]=2$

$b^2-b-\frac{2}{\alpha_{II}^2}=0 \qquad D=1+\frac{8}{\alpha_{II}^2} \qquad \boxed{b=\frac{1}{2}\left[1+\sqrt{1+\frac{8}{\alpha_{II}^2}}\right]}$

For existence of PTE we need at least one of the following two inequalities to hold:

$$\frac{\alpha_I}{2}\left[1+\sqrt{1+\frac{8}{\alpha_{II}^2}}\right]+\frac{2\alpha_I}{\alpha_{II}^2}\leq1 \qquad \frac{\alpha_I}{2\alpha_{II}^2}\left[1-\frac{\sqrt{1+8\alpha_{II}^2}-1}{4\alpha_{II}^2}\right]\geq1$$

There are cases when both $\frac{2\alpha_{II}^2}{\left[1-\frac{\sqrt{1+8\alpha_{II}^2}-1}{4\alpha_{II}^2}\right]}\leq\alpha_I$ and $\frac{2}{\alpha_I}\leq1+\sqrt{1+\frac{8}{\alpha_{II}^2}}+\frac{4}{\alpha_{II}^2}$ are not satisfied.

E.g. for $\alpha_I=1/2,\alpha_{II}=2$ both are not satisfied.

$$\left.\frac{2\alpha^2}{1-\frac{\sqrt{1+8\alpha^2}-1}{4\alpha^2}}\right|_{\alpha=2}=11.372\nleq1/2 \qquad \left.1+\sqrt{1+\frac{8}{\alpha^2}}+\frac{4}{\alpha^2}\right|_{\alpha=2}=3.7321\ngeq4$$

So, for this combination of parameters there is no equilibrium.
Hence, just adding one non-convex guy can lead to non-existence of equilibrium.
[Note: we have disproved statements from (b), (d.A) and (e).]

Exercise 2

(a) For two agents with two strategies each we have four outcomes with different pairs of utility values. For four agents with two strategies each we have sixteen outcomes with potentially different pairs of utility values. The only meaningful case is when agents play one on one, and outside actions do not affect them, but that's the case of two separate independent games, which is not what we wanted to get by replicating agents.

However for some symmetric games like "Prisoners Dilemma" or "Kitty Genovese" multiplication of agents is meaningful and gives interesting results.

(b) Same logic works for consumers with externalities. Whether new agents cast externalities on old agents and the opposite can only be determined from economic meaning and cannot be introduced as a purely mathematical concept.

The natural assumption would probably be that externalities work inside replicas and are not cast across groups. Then the number of connections is equivalent to the original problem, and the resource constraint is simply multiplied. But this situation probably won't have much economic meaning.

Exercise 3

$E^{\alpha} = \{X_i, (\succeq_i), Y_j, \omega\} \qquad \eta_j(p) = \left\{ y_j \in Y_j : py_j = \sup_{y \in Y_j} py \stackrel{def}{=} \pi_j(p) \right\}$

$w_i(p) = \alpha_i(p)\, w(p) = \alpha_i(p)\left[p\omega + \Sigma_j \pi_j(p)\right], \qquad \forall p : \{\alpha_i(p) > 0, \Sigma_i \alpha_i(p) = 1\}$

$\gamma_i(p, w_i(p)) = \{x_i \in X_i : px_i \leq w_i(p)\} \qquad \xi_i(p, w_i(p)) = \{x_i^* \in \gamma_i(p) : x_i \succeq_i \gamma_i(p)\}$

ErtWS: $\left[x_i^*, y_j^*, p^*\right]$ s.t. $\forall i : x_i^* \in \xi_i(p, w_i(p)) \qquad \forall j : y_j^* \in \eta_j(p) \qquad \Sigma_i x_i^* + \Sigma_j y_j^* = \omega$

a-b) The notion of production efficiency is the same. Support and Separation theorems follow.

First Welfare Theorem (Weak Version): ErtWS is weakly Pareto optimal.

Proof. Imagine, $x - y = \omega = x^* - y^*$ and $\forall i : x_i \succ_i x_i^*$

Then, $[x_i^* \in \xi_i(p, w_i(p)) \Rightarrow p^* x_i > p^* x_i^* \Rightarrow p^* x > p^* x^*]$, $\left[y_j^* \in \eta_j(p) \Rightarrow p^* y^* > py\right]$

$\Rightarrow p^*(x - y) > p^*(x^* - y^*)$. Contradiction ■

When we think of a ErtWS here, we fix the price vector, so both p and $\alpha_i(p)$ are constants. The answer does not depend on whether $\alpha_i(p)$ depends on p since agents take both wealth and price as given. Strong version of FWT follows if we add convexity of X and non-satiation in equilibrium. We could also use the standard FWT for a private ownership economy by setting $\alpha_i = \alpha_i(p^*)$.

c) Second Welfare Theorem states that any Pareto optimal allocation $\left[x_i^*, y_j^*\right]$ can be supported by a wealth distribution and price vector p^*, such that we attain any efficient allocation in the equilibrium. If we fix the wealth distribution as a function of price we restrict ourselves and won't be able to support all the efficient allocations.

The SWT additionally requires continuity and quasi-concavity of preferences, and convexity of the production set. We need those because we want to assure existence of equilibrium, supporting the efficient allocation.

To prove the theorem we essentially assume convexity and non-emptyness of the production set and quasi-concavity of preferences. Given that the problem is equivalent to maximization of a convex set given a convex constraint. Then we use the separating hyperplane argument to prove existence of prices.