Economics 201A, Homework #4 Anton Cheremukhin

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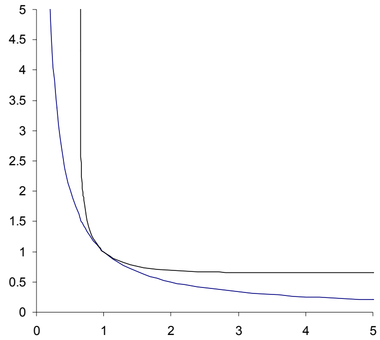
Exercise 1

(a) $v(c) = \ln c$
Bruins: $U = 0.8 \ln c_1^B + 0.2 \ln c_2^B \to \max \quad s.t.\ p_1 c_1^B + p_2 c_2^B \leq W$
Trojans: $U = 0.4 \ln c_1^T + 0.6 \ln c_2^T \to \max \quad s.t.\ p_1 c_1^T + p_2 c_2^T \leq 3W$
Demands: $c_1^B = 0.8 \frac{W}{p_1} \quad c_1^T = 0.4 \frac{3W}{p_1}$
Total supply in each state of nature: $4W$
(b) $4W = 0.8 \frac{W}{p_1} + 0.4 \frac{3W}{p_1} = 0.2 \frac{W}{p_2} + 0.6 \frac{3W}{p_2} \quad \Rightarrow \quad p_1/p_2 = 1$
(c) Caltech: $U = 0.5 \ln c_1^C + 0.5 \ln c_2^C \to \max \quad s.t.\ p_1 c_1^C + p_2 c_2^C \leq \frac{1}{2} W$
$\frac{9}{2} W = 0.8 \frac{W}{p_1} + 0.4 \frac{3W}{p_1} + 0.5 \frac{W}{2p_1} = 0.2 \frac{W}{p_2} + 0.6 \frac{3W}{p_2} + 0.5 \frac{W}{2p_2} \quad \Rightarrow \quad p_1/p_2 = 1$
(d) $8W = 0.8 \frac{W}{p_1} + 0.4 \frac{3W}{p_1} + 0.5 \frac{4W}{p_1} = 0.2 \frac{W}{p_2} + 0.6 \frac{3W}{p_2} + 0.5 \frac{4W}{p_2} \quad \Rightarrow \quad p_1/p_2 = 1$
Intuition: difference in probabilities exactly offsets the difference in wealth. Adding agents with the ratio of probabilities equal to the ratio of prices doesn't change the prices.

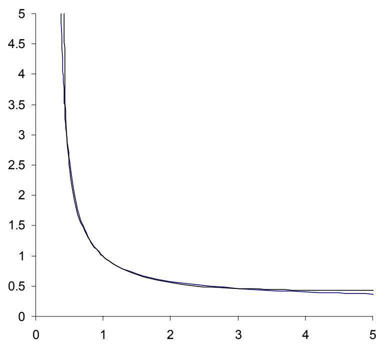
Exercise 2

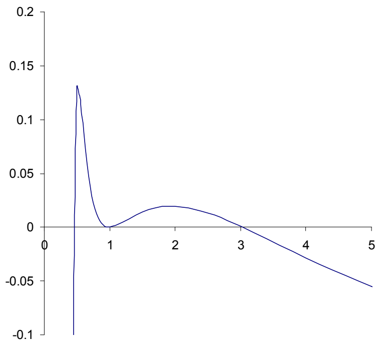
Alex: $CRRA : R(c) = -\frac{cu''(c)}{u'(c)} = const : u(c) = \frac{c^{1-R}}{1-R}, (R > 0, R \neq 1), u(c) = \ln c, (R = 1)$
Bev: $CARA : A(c) = -\frac{v''(c)}{v'(c)} = const : v(c) = -\frac{1}{A} \exp(-Ac)$
(a) Alex's $A(c)|_{c=w} = \left.\frac{R(c)}{c}\right|_{c=w} = \frac{R}{w} \qquad$ Bev's $A(c) = A$
If $\frac{R}{w} > A$, i.e. $w < \frac{R}{A}$, then Alex has greater absolute risk aversion than Bev.
(b) $\mathbf{U}^A = \pi \frac{c_1^{1-R}}{1-R} + (1-\pi) \frac{c_2^{1-R}}{1-R} \qquad \mathbf{U}^B = -\pi \frac{1}{A} \exp(-Ac_1) - (1-\pi) \frac{1}{A} \exp(-Ac_2)$
$\left.U^A\right|_{(w,w)} = \frac{w^{1-R}}{1-R} \qquad \left.U^B\right|_{(w,w)} = -\frac{1}{A} \exp(-Aw)$
Hence, the indifference curves are given by:
$\pi c_1^{1-R} + (1-\pi) c_2^{1-R} = w^{1-R}, \qquad \frac{\pi}{e^{Ac_1}} + \frac{1-\pi}{e^{Ac_2}} = \frac{1}{e^{Aw}}$
Both fuctions are continuous on R_+^2 and pass through (w, w) by construction.
Let's find their slope in (w, w):
$c_1 = \left(\frac{w^{1-R}}{\pi} - (\frac{1}{\pi} - 1) c_2^{1-R} \right)^{\frac{1}{1-R}}, \qquad c_1 = -\frac{1}{A} \log(\frac{1}{\pi} e^{-Aw} - (\frac{1}{\pi} - 1) e^{-Ac_2})$
$$\left.\frac{dc_1}{dc_2}\right|_{c_2=w} = \left.\frac{\partial(\left(\frac{w^{1-R}}{\pi} - (\frac{1}{\pi} - 1) c_2^{1-R} \right)^{\frac{1}{1-R}})}{\partial c_2}\right|_{c_2=w} = 1 - \frac{1}{\pi}$$
$$\left.\frac{dc_1}{dc_2}\right|_{c_2=w} = \left.\frac{\partial(-\frac{1}{A} \log(\frac{1}{\pi} e^{-Aw} - (\frac{1}{\pi} - 1) e^{-Ac_2}))}{\partial c_2}\right|_{c_2=w} = 1 - \frac{1}{\pi}$$
Their slope in (w, w) is the same, which means that they touch each other.
For the case A=2, R=1, W>1/2 we have proven, that Alex has higher absolute risk aversion. Hence, by proposition from the lectures, if $A_1(c)>A_2(c)$ for any c, then G_2 lies inside G_1 . Hence, the indifference curves do not intersect, and Bev's is inside Alex's:



So, the statement about two intersections is definitely not true in general.

However, the statement could be true. For example, take the case $w = 1$, $\pi = 0.5$, $A \approx 1$ and $R = A + \varepsilon > A$.The indifference curves and the difference between them look like this:





(c) By continuity, if we start shifting the endowment, we shall be getting two symmetric intersections moving away from the riskless point, but the order of the indifference curves far away from that point will remain the same. That means, that the CARA guy (Bev) will be more willing to take on smaller risks, and the CRRA guy - on bigger risks. The intuition behind that is the fact that CARA means attitude to absolute risk, measured in units of goods, while CRRA - relative risk, measured by the slope of a line, passing through the origin. So, relative risk aversion means relative to the endowment, which implies the ability to take larger absolute risk.

Exercise 3

(a) There are state claims which give a unit of good in each state. The individual is facing the following budget constraint: $p_1c^1+p_2c^2 \leq p_1+2p_2$, as the aggregate endowment is (1,2). Because the states are equally likely the maximization problem is: $\max_{\{c^A,c^B\}}[v(c^1)+v(c^2)|p_1c^1+p_2c^2 \leq p_1+2p_2]$

FOC: $\frac{v'(c^1)}{p_1} = \frac{v'(c^2)}{p_2}$ Hence, $\frac{p_2}{p_1} = \left(\frac{c^2}{c^1}\right)^{-R} = \left(\frac{c^1}{c^2}\right)^{R}$

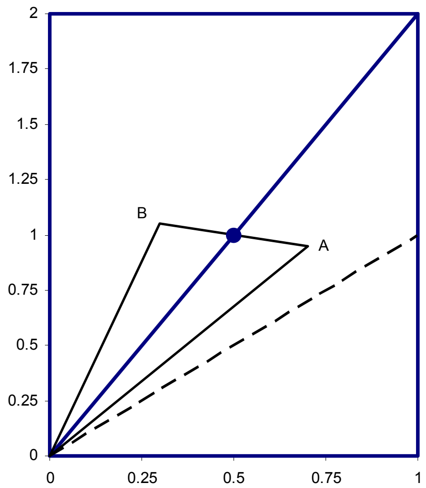
In Walrasian Equilibrium: $\frac{p_2}{p_1} = \left.\left(\frac{c^1}{c^2}\right)^{R}\right|_{(1,2)} = \frac{1}{2^R}$.

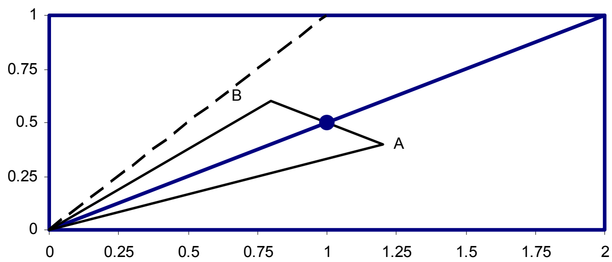
(b) Markets are complete given 2 contingent claims for 2 states of nature. Hence assets can be priced using the prices of contingent claims. Asset A gives you αp_1 and βp_2 with equal probability. Asset B gives you $(1-\alpha)p_1$ and $(2-\beta)p_2$ with equal probability. That means that

$\frac{p_A}{p_B} = \frac{\alpha p_1+\beta p_2}{(1-\alpha)p_1+(2-\beta)p_2} = \frac{\alpha+\beta\frac{p_2}{p_1}}{(1-\alpha)+(2-\beta)\frac{p_2}{p_1}} = \frac{2^R\alpha+\beta}{2^R(1-\alpha)+(2-\beta)}$

(c) The price of asset A relative to the price of asset B will rise as R rises if:
$\frac{\partial}{\partial R}\frac{2^R\alpha+\beta}{2^R(1-\alpha)+(2-\beta)} = \frac{2^R\ln 2}{((1-\alpha)2^R+2-\beta)^2}[2\alpha-\beta] > 0 \qquad$ i.e. $2\alpha > \beta$.

(d) Intuition: The PO-set is the diagonal of the Engeworth box because agents have identical homothetic preferences. The condition $2\alpha > \beta$ means that the point, representing asset A, must be below the diagonal in the Edgeworth Box. The assets sum up to the whole endowment. Hence, the point, representing asset B, must be above the diagonal in the Edgeworth Box. In this case the relative price of the asset nearer to the riskless line has to grow as risk aversion grows to keep equilibrium on the diagonal. This result also depends on the fact that aggregate endoment of good 2 is bigger. In the opposite case the other asset would be nearer to the riskless line. The intuition is demonstrated in the following graphs.





(e) There are two assets: A and B. Asset A gives α in state 1 and β in state 2. Asset B gives $1-\alpha$ in state 1 and $2-\beta$ in state 2. If assets are traded, the individual is facing the following budget constraint: $p_A c_A + p_B c_B \leq p_A w_A + p_B w_B$. His endowment should be expressed in terms of assets. The agent owns everything if he has equal amounts of both assets equal to 1. The payoff in state 1 is $\alpha c_A + (1-\alpha)c_B$. The payoff in state 2 is $\beta c_A + (2-\beta)c_B$. Because the states are equally likely the maximization problem is:

$$\max_{\{c^A,c^B\}}[v(\alpha c^A + (1-\alpha)c^B) + v(\beta c^A + (2-\beta)c^B) | p_A c^A + p_B c^B \leq p_A + p_B]$$

FOC: $\quad \alpha v'(\alpha c^A + (1-\alpha)c^B) + \beta v'(\beta c^A + (1-\beta)c^B) = \lambda p_A$

$\qquad (1-\alpha)v'(\alpha c^A + (1-\alpha)c^B) + (2-\beta)v'(\beta c^A + (2-\beta)c^B) = \lambda p_B$

Hence, $\frac{p_B}{p_A} = \left.\frac{(1-\alpha)\left(\frac{\alpha c^A+(1-\alpha)c^B}{\beta c^A+(2-\beta)c^B}\right)^{-R}+(2-\beta)}{\alpha\left(\frac{\alpha c^A+(1-\alpha)c^B}{\beta c^A+(2-\beta)c^B}\right)^{-R}+\beta}\right|_{(c^A=c^B)} = \frac{2^R(1-\alpha)+(2-\beta)}{2^R\alpha+\beta}.$

Given that $\frac{\alpha}{1-\alpha} \neq \frac{\beta}{2-\beta}$ the price vector is the same as in part (b) because payoffs of the shares are independent and hence markets are complete.

Exercise 4

(a) There are four states. There are state claims which give a unit of good in each state. The aggregate endowment is $(1, 4, 9, 16)$. Because the states are equally likely the maximization problem is: $\max_{\{c^A, c^B\}}[v(c^1) + v(c^2) + v(c^3) + v(c^4)|p_1c^1 + p_2c^2 + p_3c^3 + p_4c^4 \leq p_1 + 4p_2 + 9p_3 + 16p_4]$

FOC: $\frac{v'(c^i)}{p_i} = \lambda$ Hence, $\frac{p_i}{p_j} = \left(\frac{c^i}{c^j}\right)^{-1/2} = \sqrt{\frac{c^j}{c^i}}$.

From the the representative-agent intuition we get the following price-vector:

$(p_1, p_2, p_3, p_4) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4})$

From the first order conditions the consumption of claims is the same function of prices for each agent: $\frac{c^j}{c^i} = \left(\frac{p_i}{p_j}\right)^2$. Hence, the consumptions of all agents are linearly dependent. As long as they sum up to the whole endowment, this means, that every agent is consuming a fraction of the endowment.

(b) Using prices of contingent claims the prices for the assets are:

$p_B = z_B p = 0 + \frac{3}{2} + \frac{8}{3} + \frac{15}{4} = \frac{95}{12}$ $\quad p_A = z_A p = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$ $\quad \frac{p_B}{p_A} = \frac{19}{5}$

(c) Here the markets are incomplete. Hence, we have to solve the whole problem.

Now there are only two assets: A and B. Asset A gives (1,1,1,1). Asset B gives (0,3,8,15). Because the initial endowment of goods is represented by a unique linear combination of these two assets (1,1), the endowments of assets A and B are (1,1). The individual is facing the budget constraint: $p_A c_A + p_B c_B \leq p_A + p_B$. Because the states are equally likely the maximization problem is:

$\max_{\{c^A, c^B\}}[\Sigma_{s=1}^4 v(z_s^A c^A + z_s^B c^B)|p_A c^A + p_B c^B \leq p_A + p_B]$

FOC: $\Sigma_{s=1}^4 \frac{z_s^A}{\sqrt{z_s^A c^A + z_s^B c^B}} = \lambda p_A$ $\quad \Sigma_{s=1}^4 \frac{z_s^B}{\sqrt{z_s^A c^A + z_s^B c^B}} = \lambda p_B$

Hence, $$\frac{p_B}{p_A} = \left.\frac{\Sigma_{s=1}^4 \frac{z_s^B}{\sqrt{z_s^A c^A + z_s^B c^B}}}{\Sigma_{s=1}^4 \frac{z_s^A}{\sqrt{z_s^A c^A + z_s^B c^B}}}\right|_{(1,1)} = \frac{\Sigma_{s=1}^4 \frac{z_s^B}{\sqrt{z_s^A + z_s^B}}}{\Sigma_{s=1}^4 \frac{z_s^A}{\sqrt{z_s^A + z_s^B}}} = \frac{\frac{0}{\sqrt{1+0}} + \frac{3}{\sqrt{1+3}} + \frac{8}{\sqrt{1+8}} + \frac{15}{\sqrt{1+15}}}{\frac{1}{\sqrt{1+0}} + \frac{1}{\sqrt{1+3}} + \frac{1}{\sqrt{1+8}} + \frac{1}{\sqrt{1+15}}} = \frac{19}{5}$$

This is the same as with full markets. Hence the equilibrium price ratio of part (b) remains.

The result totally depends on whether the aggregate endowment can be expressed in terms of existing assets. If this was not true, one would generally get a corner solution, where not all the endowment of goods can be consumed, and hence some of it has to be "freely disposed". In this case the answer would differ from the complete-markets case.

Example: Asset A gives (1,2,3,4). Asset B gives (5,4,3,2).

In this case we first have to solve for the best endowment of these assets, so that we get maximum utility, constrained by the fact, that endowment in each state is no bigger that the actual endoment of good in that state:

$\max_{\{w^A,w^B\}}[\Sigma_{s=1}^4 v(z_s^A w^A + z_s^B w^B)|z_i^A w^A + z_i^B w^B \leq w_i]$

If the number of states (S) is bigger than the number of assets (N), that normally N constraints will be binding and S-N constraints won't. For our numerical example this leads to the case when the first and fourth constraints are binding:

Constraints: $w^A + 5w^B \leq 1$ $\quad 2w^A + 4w^B \leq 4$ $\quad 3w^A + 3w^B \leq 9$ $\quad 4w^A + 2w^B \leq 16$

Case:1,2 $\{w^A + 5w^B = 1, 2w^A + 4w^B = 4\}$, Solution is: $\left[w^A = \frac{8}{3}, w^B = -\frac{1}{3}\right]$

$$\sqrt{w^A + 5w^B} + \sqrt{2w^A + 4w^B} + \sqrt{3w^A + 3w^B} + \sqrt{4w^A + 2w^B}\Big|_{w^A=\frac{8}{3},w^B=-\frac{1}{3}} = 8.808$$

Case:1,3 $\{w^A + 5w^B = 1, 3w^A + 3w^B = 9\}$, Solution is: $\left[w^A = \frac{7}{2}, w^B = -\frac{1}{2}\right]$

$$\sqrt{w^A + 5w^B} + \sqrt{2w^A + 4w^B} + \sqrt{3w^A + 3w^B} + \sqrt{4w^A + 2w^B}\Big|_{w^A=\frac{7}{2},w^B=-\frac{1}{2}} = 9.8416$$

Case:1,4 $\{w^A + 5w^B = 1, 4w^A + 2w^B = 16\}$, Solution is: $\left[w^A = \frac{13}{3}, w^B = -\frac{2}{3}\right]$

$$\sqrt{w^A + 5w^B} + \sqrt{2w^A + 4w^B} + \sqrt{3w^A + 3w^B} + \sqrt{4w^A + 2w^B}\Big|_{w^A=\frac{13}{3},w^B=-\frac{2}{3}} = 10.766$$

Case:2,3 $\left\{2w^A + 4w^B = 4, 3w^A + 3w^B = 9\right\}$, Solution is: $\left[w^A = 4, w^B = -1\right]$

$$\sqrt{w^A + 5w^B} + \sqrt{2w^A + 4w^B} + \sqrt{3w^A + 3w^B} + \sqrt{4w^A + 2w^B}\Big|_{w^A=4,w^B=-1} = \varnothing$$

Case:2,4 $\left\{2w^A + 4w^B = 4, 4w^A + 2w^B = 16\right\}$, Solution is: $\left[w^A = \frac{14}{3}, w^B = -\frac{4}{3}\right]$

$$\sqrt{w^A + 5w^B} + \sqrt{2w^A + 4w^B} + \sqrt{3w^A + 3w^B} + \sqrt{4w^A + 2w^B}\Big|_{w^A=\frac{14}{3},w^B=-\frac{4}{3}} = \varnothing$$

Case:3,4 $\left\{3w^A + 3w^B = 9, 4w^A + 2w^B = 16\right\}$, Solution is: $\left[w^A = 5, w^B = -2\right]$

$$\sqrt{w^A + 5w^B} + \sqrt{2w^A + 4w^B} + \sqrt{3w^A + 3w^B} + \sqrt{4w^A + 2w^B}\Big|_{w^A=5,w^B=-2} = \varnothing$$

So, the solution is: $w^A = \frac{13}{3}, w^B = -\frac{2}{3}$. The incomplete-market price vector will be:

$$\frac{p_B}{p_A} = \frac{\frac{1}{\sqrt{1*\frac{13}{3}+5*\left(-\frac{2}{3}\right)}}+\frac{2}{\sqrt{2*\frac{13}{3}+4*\left(-\frac{2}{3}\right)}}+\frac{3}{\sqrt{3*\frac{13}{3}+3*\left(-\frac{2}{3}\right)}}+\frac{4}{\sqrt{4*\frac{13}{3}+2*\left(-\frac{2}{3}\right)}}}{\frac{5}{\sqrt{1*\frac{13}{3}+5*\left(-\frac{2}{3}\right)}}+\frac{4}{\sqrt{2*\frac{13}{3}+4*\left(-\frac{2}{3}\right)}}+\frac{3}{\sqrt{3*\frac{13}{3}+3*\left(-\frac{2}{3}\right)}}+\frac{2}{\sqrt{4*\frac{13}{3}+2*\left(-\frac{2}{3}\right)}}} = \frac{\frac{1}{3}\sqrt{6}+\frac{3}{11}\sqrt{11}+2}{\frac{2}{3}\sqrt{6}+\frac{3}{11}\sqrt{11}+\frac{11}{2}} = 0.46296$$

However, in the presence of contingent claims the relative price is:

$(p_1, p_2, p_3, p_4) = (1, 1, 1, 1)$ $\quad \frac{p_B}{p_A} = \frac{1+2+3+4}{5+4+3+2} = \frac{5}{7} = 0.71429$