Economics 201A, Homework #2 Anton Cheremukhin

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Exercise 1

(a) $U(x) = \sqrt[2]{x_1 x_2} + \delta \sqrt[3]{x_3^2 x_4}$

Preferences are homothetic if and only if $\forall x \forall \lambda > 0 \exists \mu > 0 : \nabla U(\lambda x) = \mu \nabla U(x)$

$$\nabla U(\lambda x) = \begin{bmatrix} \frac{1}{2}(\frac{\lambda x_2}{\lambda x_1})^{1/2} \\ \frac{1}{2}(\frac{\lambda x_1}{\lambda x_2})^{1/2} \\ \delta\frac{2}{3}(\frac{\lambda x_4}{\lambda x_3})^{1/3} \\ \delta\frac{1}{3}(\frac{\lambda x_3}{\lambda x_4})^{2/3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\frac{x_2}{x_1})^{1/2} \\ \frac{1}{2}(\frac{x_1}{x_2})^{1/2} \\ \delta\frac{2}{3}(\frac{x_4}{x_3})^{1/3} \\ \delta\frac{1}{3}(\frac{x_3}{x_4})^{2/3} \end{bmatrix} = \nabla U(x) \qquad \Rightarrow \qquad \mu = 1$$

So given preferences are homothetic.

For agents with identical homothetic preferences (but possibly different endowments), aggregate excess demand can be derived from maximization of a utility function of a representative agent whose endowment is the sum of individual endowments. Such an economy has unique equilibrium.

Endowment: $w = (a, 4a, b, b), a > 0, b > 0$, taking $a = 100, b = 200$ as an example.

Taking p_1 as the numeraire use the FOCs: $\frac{\partial U(x)/\partial x_i}{\partial U(x)/\partial x_j} = \frac{p_i}{p_j}$

So $p_2 = \frac{\partial U(x)/\partial x_2}{\partial U(x)/\partial x_1} = \frac{\frac{1}{2}(\frac{w_1}{w_2})^{1/2}}{\frac{1}{2}(\frac{w_2}{w_1})^{1/2}} = \frac{a}{4a} = \frac{1}{4}$,

$p_3 = \frac{\partial U(x)/\partial x_3}{\partial U(x)/\partial x_1} = \frac{\delta\frac{2}{3}(\frac{x_4}{x_3})^{1/3}}{\frac{1}{2}(\frac{w_2}{w_1})^{1/2}} = \frac{\delta\frac{2}{3}(\frac{b}{b})^{1/3}}{\frac{1}{2}(\frac{4a}{a})^{1/2}} = \frac{2\delta}{3}$,

$p_4 = \frac{\partial U(x)/\partial x_1}{\partial U(x)/\partial x_4} = \frac{\frac{1}{2}(\frac{w_2}{w_1})^{1/2}}{\delta\frac{1}{3}(\frac{x_3}{x_4})^{2/3}} = \frac{\delta\frac{1}{3}(\frac{b}{b})^{2/3}}{\frac{1}{2}(\frac{4a}{a})^{1/2}} = \frac{\delta}{3}$

The equilibrium price vector is $(1, \frac{1}{4}, \frac{2\delta}{3}, \frac{\delta}{3})$.

(b) $U(x) = \sqrt[2]{x_{11} x_{12}} + \delta \sqrt[3]{x_{21}^2 x_{22}}$, (1st-time,2nd-good)

The problem does not differ mathematically, so the prices are: $(1, \frac{1}{4}, \frac{1}{3}, \frac{1}{6})$.

(c) This price vector could be reformulated $(p_{11}, p_{12}, p_{21}, p_{22}) = (p_1, p_{12}, \frac{p_1}{1+r}, \frac{p_{22}}{1+r})$

This is still 4 variables, one of which could be normalized: $(1, \frac{1}{4}, \frac{1}{1+2}, \frac{1}{(1+2)2})$

So the equilibrium interest rate is 200%, todays and tomorrows price vectors are $(1, \frac{1}{4})$ and $(1, \frac{1}{2})$ respectively.

(d) Imagine we have 1 apple and 4 peaches today and 6 of both fruit tomorrow. The interesting thing is that we can store them for tomorrow, but can't eat tomorrow's fruit today. Not to store them we need MU_1 be no less than MU_3 and MU_2 be no less than MU_4, denoting MU_i the marginal utility of x_j. So, $\frac{1}{2}(\frac{4a}{a})^{1/2} \geq \delta\frac{2}{3}(\frac{6b}{6b})^{1/3}$, $\frac{1}{2}(\frac{a}{4a})^{1/2} \geq \delta\frac{1}{3}(\frac{6b}{6b})^{2/3}$, which implies $(\delta \leq 3/2) \cap (\delta \leq 3/4)$. That means, that for $\delta \leq 3/4$ the equilibrium remains the same given the possibility of storage.

Exercise 2

(a) For one unit of money you can buy $1/p_1$units today and sell it tomorrow, getting p_2/p_1.Either way you could put it in a bank and get $(1+r)$ in return. If it's an equilibrium than these amounts should be equal:$(1+r) = p_2/p_1$. If firms are price-takers, they can't change anything.

(b) For $p_1(q_1) = b_1/q_1^{1/e}, p_2(q_2) = b_2/q_2^{1/e}$ we get: $(1+r) = \frac{b_2}{q_2^{1/e}}\frac{q_1^{1/e}}{b_1} = \frac{b_2}{(Q-q_1)^{1/e}}\frac{q_1^{1/e}}{b_1}$
$\frac{Q}{q_1} = \left[\frac{1}{(1+r)}\frac{b_2}{b_1}\right]^e + 1$, Hence $q_1 = \frac{Q}{\left[\frac{1}{(1+r)}\frac{b_2}{b_1}\right]^e+1}$, $q_2 = \frac{Q}{\left[(1+r)\frac{b_1}{b_2}\right]^e+1}$.

(c) Profit is given by $\pi = q_1^{1-1/e}b_1 + (Q-q_1)^{1-1/e}\frac{b_2}{1+r}$, which is maximized with respect to q_1. This implies $q_1^{-1/e}b_1 = (Q-q_1)^{-1/e}b_2/(1+r)$, which transforms to $q_1 = \frac{Q}{\left[\frac{b_2}{b_1}\frac{1}{1+r}\right]^e+1}$.
Hence, $q_2 = \frac{Q}{\left[(1+r)\frac{b_1}{b_2}\right]^e+1}$, $\quad p_1 = \frac{b_1}{\left[\frac{Q}{\left[\frac{b_2}{b_1}\frac{1}{1+r}\right]^e+1}\right]^{1/e}}$, $\quad p_2 = \frac{b_1}{\left[\frac{Q}{\left[(1+r)\frac{b_1}{b_2}\right]^e+1}\right]^{1/e}}$.

(d) It is not a surprizing result, but this is an occasion!
Thinking of the problem more generally:

$\pi = (1+r)p_1(q_1)q_1 + p_2(q_2)q_2 - (1+r)c_1(q_1) - c_2(q_2)$
FOC: $\quad \pi_1' = (1+r)\left[p_1'(q_1)q_1 + p_1(q_1) - c_1'(q_1)\right] = 0$

$$\pi_2' = p_2'(q_2)q_2 + p_2(q_2) - c_2'(q_2) = 0$$

This implies, that for the answer to satisfy the no-arbitrage condition $p_2 = p_1(1+r)$, we need the following conditions to hold:
1) $q_1 + q_2 = Q$, i.e. there is an upper bound on the sum of output

2) $(1+r)c_1'(q_1) = c_2'(q_2)$, i.e. marginal costs are equal (or absent)

3) $\frac{p_1'(q_1)q_1}{p_1(q_1)} = \frac{p_2'(q_2)q_2}{p_2(q_2)}$, i.e. the elasticities of the demand curves are equal

Loosely speaking, it should be an endowment economy with equal elasticities of demands.
It's impossible for such a thing to happen in real life, so it's an occasion.

Exercise 3

$p(q,\theta) = a - \frac{q}{\theta}, \qquad C(q,\theta) = \theta q$

(a) $\pi(q,\theta) = pq - c(q) = (a - \frac{q}{\theta})q - \theta q$

$\pi' = \begin{bmatrix} a - 2\frac{q}{\theta} - \theta \\ \frac{q^2}{\theta^2} - q \end{bmatrix}, \qquad \pi'' = \begin{bmatrix} -\frac{2}{\theta} & \frac{2q}{\theta^2} - 1 \\ \frac{2q}{\theta^2} - 1 & -\frac{2q^2}{\theta^3} \end{bmatrix}$

For the profit function to be concave we need the Hessian to be negative semi-definite.

That implies 1) $(-\frac{2}{\theta})(-\frac{2q^2}{\theta^3}) - (\frac{2q}{\theta^2} - 1)^2 = \left(\frac{4q}{\theta^2} - 1\right) \geq 0$, i.e. $q \geq \left(\frac{\theta}{2}\right)^2$

and 2) $-\frac{2}{\theta} < 0$, i.e. $\theta > 0$.

(b) $\pi'_q = 0$ implies $q(\theta) = (a-\theta)\frac{\theta}{2}$

$\pi'_\theta = 0$ implies $\theta(q) = q^{1/2}$

(c) We could either plug one into the profit function or intersect them.

1) $\pi(\theta, q(\theta)) = (a - \frac{(a-\theta)}{\theta}\frac{\theta}{2})(a-\theta)\frac{\theta}{2} - \theta(a-\theta)\frac{\theta}{2} = \theta\left[\frac{a-\theta}{2}\right]^2$

$\frac{d}{d\theta}\pi(\theta, q(\theta)) = \left[\frac{a-\theta}{2}\right]^2 - 2\theta\frac{1}{2}\left[\frac{a-\theta}{2}\right] = 0 \Rightarrow \theta^* = \frac{a}{3}, \qquad q(\theta^*) = \frac{a^2}{9}$

2) $(a-\theta)\frac{\theta}{2} = \theta^2$ implies $\theta^* = \frac{a}{3}$, $q(\theta^*) = \frac{a^2}{9}$.

(d) $p(q,\beta) = a\beta - q, \qquad C(q,\beta) = \beta^2 q^2$

$\pi = (a\beta - q)q - \beta^2 q^2,$

$\pi' = a\beta - 2q - 2\beta^2 q = 0, \qquad \pi' = aq - 2\beta q^2 = 0$

Using the plug-in method: $q(\beta) = \frac{a}{2}\frac{\beta}{1+\beta^2}$,

$\pi(q(\beta),\beta) = \frac{a^2}{4}\frac{\beta^2}{1+\beta^2}, \qquad \pi'_\beta = \frac{a^2}{4}\frac{1}{(1+\beta^2)^2} > 0$

This one solves only for $a = 0$ at $q^* = 0$, β^* any.

Otherwise no solution, as profit grows forever when $\beta \to \infty$.

If there is an upper bound on β, then $\beta^* = \overline{\beta}$, $q^* = \frac{a}{2}\frac{\overline{\beta}}{1+\overline{\beta}^2}$.

Exercise 4

$p_1 = 605 - 2q_1^2, \qquad p_2 = 80 - 2q_2, \qquad p_3 = 70 - 2q_3, \qquad C(q) = 5(q_1 + q_2 + q_3) + 20q_0$

(a) $\max[(605 - 2q_1^2)q_1 + (80 - 2q_2)q_2 + (70 - 2q_3)q_3 - 5(q_1 + q_2 + q_3) - 20q_0|\ q_1 \geq 0, q_2 \geq 0, q_3 \geq 0, q_0 \geq \max(q_1, q_2, q_3)]$

(b) We can be sure that the last condition works as an equality, as there is no reason for paying more. So we should just solve for seven cases
($q_2, q_3 < q_1 = q_0$; $q_1, q_3 < q_2 = q_0$; $q_1, q_2 < q_3 = q_0$; $q_1 < q_2 = q_3 = q_0$; $q_2 < q_1 = q_3 = q_0$; $q_3 < q_1 = q_2 = q_0$; $q_1 = q_2 = q_3 = q_0$) and then check conditions. If more than 1 solution remains, then we shall compare the profits.

Comparing the maximum q_1 and the minimum q_2 and q_3 (when costs payed are at maximum or at minimum):
$605 - 6q_1^2 - 5 = 0 \qquad \Rightarrow q_1 = 10$
$80 - 4q_2 - 25 = 0 \qquad \Rightarrow q_2 = 13.75$
$70 - 4q_3 - 25 = 0 \qquad \Rightarrow q_3 = 11.25$

This means that the upper bound of q_1 is less than the lower bounds of q_2 and q_3. Hence, q_1 will never be the maximum, and only three cases remain:
($q_1, q_3 < q_2 = q_0$; $q_1, q_2 < q_3 = q_0$; $q_1 < q_2 = q_3 = q_0$)

1) $q_1, q_3 < q_2 = q_0$
FOC: $80 - 4q_2 - 25 \leq 0$, with $=$ if $q_2 \geq 0$
$70 - 4q_3 - 5 \leq 0$, with $=$ if $q_3 \geq 0$
$q_2 = 55/4 \simeq 13.75, \qquad q_3 = 65/4 \simeq 16.25, \qquad q_1 = 10 \qquad q_2$ is not the maximum

2) $q_1, q_2 < q_3 = q_0$
FOC: $80 - 4q_2 - 5 \leq 0$, with $=$ if $q_2 \geq 0$
$70 - 4q_3 - 25 \leq 0$, with $=$ if $q_3 \geq 0$
$q_2 = 75/4 = 18.75, \qquad q_3 = 45/4 \simeq 11.25, \qquad q_1 = 10 \qquad q_3$ is not the maximum

3) $q_0 = q_2 = q_3 > q_1$
FOC: $80 - 4q_0 + 70 - 4q_0 - 5 - 5 - 20 \leq 0$, with $=$ if $q_0 \geq 0$
$q_2 = q_3 = q_0 = 120/8 = 15, \qquad q_1 = 10$
$p_1 = 405, \qquad p_2 = 50, \qquad p_3 = 40.$ That is the answer.

(c) $SS_1 = \int_0^{q_1}(605 - 2q^2)dq = 605q_1 - \frac{2}{3}q_1^3$,
$SS_2 = \int_0^{q_2}(80 - 2q)dq = 80q_2 - q_2^2, \qquad SS_3 = \int_0^{q_3}(70 - 2q)dq = 70q_3 - q_3^2.$
$SS = \sum(SS_j - 5q_j) - 20 = 600q_1 - \frac{2}{3}q_1^3 + 75q_2 - q_2^2 + 65q_3 - q_3^2 - 20\max[q_1, q_2, q_3]$

The logic is the same. The single-equal variants are: $q_1 \in \{\sqrt{600/2}, \sqrt{580/2}\} \simeq \{17.32, 17.03\}$, $q_2 \in \{75/2, 55/2\} \simeq \{37.5, 27.5\}$, $q_3 \in \{65/2, 45/2\} \simeq \{32.5, 22.5\}$

So we have a similar situation again: $q_0 = q_2 = q_3 > q_1$
FOC: $75 - 2q_0 + 65 - 2q_0 - 20 \leq 0$, with $=$ if $q_0 \geq 0$
$q_2 = q_3 = q_0 = 120/4 = 30, \qquad q_1 = 10\sqrt{3} \simeq 17.3$
$p_1 = 5, \qquad p_2 = 20, \qquad p_3 = 10.$ That is the answer. The profit of the firm is still positive.