ECON201A HW#1

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Problem 1

(a) $\max\{x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}|x_1p_1+x_2p_2+x_3p_3\le I, x_i\ge 0\}$
$\pounds = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}+\lambda(I-x_1p_1-x_2p_2-x_3p_3)$
First order conditions:
$\frac{\partial \pounds}{\partial x_i} = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\cdot\frac{\alpha_i}{x_i}-\lambda p_i = 0 \ \cap \ x_1p_1+x_2p_2+x_3p_3=I$
The solution is straightforward $p_ix_i=\alpha_i\frac{U}{\lambda}\Rightarrow\frac{U}{\lambda}(\alpha_1+\alpha_2+\alpha_3)=I$
Demand functions: $x_i(p,I)=\frac{\alpha_i}{\Sigma\alpha_i}\frac{I}{p_i}$, for all $i=1,2,3$
Maximized utility: $v(p,I)=\prod\left(\frac{\alpha_i}{\Sigma\alpha_i}\frac{I}{p_i}\right)^{\alpha_i}=\left(\frac{I}{\Sigma\alpha_i}\right)^{\Sigma\alpha_i}\prod\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}$
(b)$-\max\{-(z_1r_1+z_2r_2+z_3r_3)|z_1^{\alpha_1}z_2^{\alpha_2}z_3^{\alpha_3}\ge q, z_i\ge 0\}$
$\pounds=-(z_1r_1+z_2r_2+z_3r_3)+\lambda(z_1^{\alpha_1}z_2^{\alpha_2}z_3^{\alpha_3}-q)$
Production function is strictly concave, so
$\frac{\partial\pounds}{\partial z_i}=-r_i+\lambda z_1^{\alpha_1}z_2^{\alpha_2}z_3^{\alpha_3}\cdot\frac{\alpha_i}{z_i}=0\ \cap\ z_1^{\alpha_1}z_2^{\alpha_2}z_3^{\alpha_3}=q$
Use of inputs: $z_i=\alpha_iq\lambda/r_i$
$q=z_1^{\alpha_1}z_2^{\alpha_2}z_3^{\alpha_3}=(q\lambda)^{\Sigma\alpha_i}\prod\left(\frac{\alpha_i}{r_i}\right)^{\alpha_i}, \lambda=q^{(1/\Sigma\alpha_i-1)}\prod\left(\frac{r_i}{\alpha_i}\right)^{\alpha_i/\Sigma\alpha_i}$
Cost function: $C(q)=q\lambda\Sigma\alpha_i=\Sigma\alpha_i(q\prod\left(\frac{r_i}{\alpha_i}\right)^{\alpha_i})^{1/\Sigma\alpha_i}$

Problem 2

(a) $\max\{(a+x_1)^{\alpha_1}x_2^{\alpha_2}|x_1p_1+x_2p_2\le I, x_i\ge 0\}$
$\pounds=(a+x_1)^{\alpha_1}x_2^{\alpha_2}+\lambda(I-x_1p_1-x_2p_2)$
First order conditions :
$\frac{\partial\pounds}{\partial x_1}=(a+x_1)^{\alpha_1}x_2^{\alpha_2}\cdot\frac{\alpha_1}{x_1+a}-\lambda p_1=0$
$\frac{\partial\pounds}{\partial x_2}=(a+x_1)^{\alpha_1}x_2^{\alpha_2}\cdot\frac{\alpha_2}{x_2}-\lambda p_2=0 \qquad x_1p_1+x_2p_2=I$
Rearranging gives $p_1(x_1+a)=\alpha_1\frac{U}{\lambda},\ p_2x_2=\alpha_2\frac{U}{\lambda}$
$\Rightarrow\frac{U}{\lambda}(\alpha_1+\alpha_2)=I+ap_1$
Demand functions:$\{\ x_1(p,I)=\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{I+ap_1}{p_1}-a,\ x_2(p,I)=\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{I+ap_1}{p_2}\}$
To make sure that demand is positive rearrange:
$\alpha_1I+\alpha_1ap_1-\alpha_1p_1a-\alpha_2p_1a=I\alpha_1-ap_1\alpha_2\ge 0\Leftrightarrow\frac{I}{p_1}\ge\frac{a\alpha_2}{\alpha_1}$
In case $\frac{I}{p_1}<\frac{a\alpha_2}{\alpha_1}$ we get a corner solution $\{x_1=0,x_2=\frac{I}{p_2}\}$
One can summarize it to: $x_1(p,I)=\frac{1}{\alpha_1+\alpha_2}[\frac{\alpha_1I}{p_1}-\alpha_2a]_+$
$x_2(p,I)=\frac{1}{p_2}(I-\frac{1}{\alpha_1+\alpha_2}[\alpha_1I-\alpha_2ap_1]_+)$
Maximized utility: $v(p,I)=\left(\frac{I+ap_1}{\Sigma\alpha_i}\right)^{\Sigma\alpha_i}\prod\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}$ if $\frac{I}{p_1}\ge\frac{a\alpha_2}{\alpha_1}$
$\frac{I}{p_2}^{\alpha_2}a^{\alpha_1}$ otherwise
(b) $\max\{(a+x_0+x_1)^{\alpha_1}x_2^{\alpha_2}|x_1p_1+x_2p_2+x_0p_0\le I, x_i\ge 0, x_0=\{0,1\}\}$
$\pounds=(a+x_0+x_1)^{\alpha_1}x_2^{\alpha_2}+\lambda(I-x_1p_1-x_2p_2-x_0p_0)$
Need to solve for the case $x_0=1$. Similar to (a):
$\frac{\partial\pounds}{\partial x_1}=(a+1+x_1)^{\alpha_1}x_2^{\alpha_2}\cdot\frac{\alpha_1}{x_1+a+1}-\lambda p_1=0$

$\frac{\partial \pounds}{\partial x_2} = (a+1+x_1)^{\alpha_1} x_2^{\alpha_2} \cdot \frac{\alpha_2}{x_2} - \lambda p_2 = 0 \qquad x_1 p_1 + x_2 p_2 + p_0 = I$

$p_1(x_1 + a + 1) = \alpha_1 \frac{U}{\lambda}, p_2 x_2 = \alpha_2 \frac{U}{\lambda} \Rightarrow \frac{U}{\lambda}(\alpha_1 + \alpha_2) = I + ap_1 + p_1$

Demand functions: $x_1(p, I) = \frac{\alpha_1}{\alpha_1+\alpha_2} \frac{I - p_0 + (a+1)p_1}{p_1} - (a+1)$,

$$x_2(p, I) = \frac{\alpha_2}{\alpha_1+\alpha_2} \frac{I - p_0 + (a+1)p_1}{p_2}$$

Maximized utility: $\quad v(p, I) = \left(\frac{I - p_0 + (a+1)p_1}{\Sigma\alpha_i}\right)^{\Sigma\alpha_i} \prod \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}$

One can easily see that $x_0 = 1$ is prefered to $x_0 = 0$ when $p_1 \geq p_0$.

So the general maximized utility is:

$$v(p, I) = \left(\frac{I + ap_1 + \max\{p_1 - p_0, 0\}}{\Sigma\alpha_i}\right)^{\Sigma\alpha_i} \prod \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}$$

(c) Demand functions do not take into account the fact that income can be less than p_0 and/or $ap_1 \frac{\alpha_2}{\alpha_1}$ or less than $p_0 + p_1(a+1)\frac{\alpha_2}{\alpha_1}$.

To check for this we need to look at the four types of solutions.

Let us make some assignments first. Define demands and values:

$B_1 = \frac{I}{p_2}$, $B_2 = \frac{\alpha_2}{\alpha_1+\alpha_2} \frac{I + ap_1}{p_2}$, $B_3 = \frac{I - p_0}{p_2}$, $B_4 = \frac{\alpha_2}{\alpha_1+\alpha_2} \frac{I - p_0 + (a+1)p_1}{p_2}$,

$A_2 = \frac{\alpha_1}{\alpha_1+\alpha_2} \frac{I + ap_1}{p_1} - a$, $A_4 = \frac{\alpha_1}{\alpha_1+\alpha_2} \frac{I - p_0 + (a+1)p_1}{p_1} - (a+1)$

$V_1 = (\frac{I}{p_2})^{\alpha_2} a^{\alpha_1}$, $V_2 = \left(\frac{I + ap_1}{\Sigma\alpha_i}\right)^{\Sigma\alpha_i} \prod \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}$,

$V_3 = (\frac{I - p_0}{p_2})^{\alpha_2} (a+1)^{\alpha_1}$, $V_4 = \left(\frac{I - p_0 + (a+1)p_1}{\Sigma\alpha_i}\right)^{\Sigma\alpha_i} \prod \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}$.

Define conditions:

$C_1 : I \geq p_0$, $C_2 : I \geq ap_1 \frac{\alpha_2}{\alpha_1}$, $C_3 : I \geq (a+1)p_1 \frac{\alpha_2}{\alpha_1} + p_0$,

$C_4 : V_3 \geq V_1$, $C_5 : V_4 \geq V_2$, $C_6 : V_3 \geq V_2$.

Always true: $V_2 \geq V_1$, $\quad V_4 \geq V_3$, $\quad C_3 \Rightarrow C_1 and\ C_2$, $\quad C_6 \Rightarrow C_5$.

Solutions:

$v()$	x_0	x_1	x_2
V_1	0	0	B_1
V_2	0	A_2	B_2
V_3	1	0	B_3
V_4	1	A_4	B_4

Cases:

C_1	C_2	C_3	C_4	C_5	C_6	sol
+	+	+	∀	+	∀	4
+	+	+	∀	∀	−	2
+	+	−	+	∀	+	3
+	+	−	∀	∀	−	2
+	−	−	+	∀	∀	3
+	−	−	−	∀	∀	1
−	+	−	∀	∀	∀	2
−	−	−	∀	∀	∀	1

Summarizing cases gives:

- $I < \min(p_0, \frac{ap_1\alpha_2}{\alpha_1})$ *or* $p_0 \leq I < \frac{ap_1\alpha_2}{\alpha_1}, V_1 > V_3 \Rightarrow$ *solution* 1
- $I \geq \max(p_0, \frac{ap_1\alpha_2}{\alpha_1}), I < p_0 + p_1(a+1)\frac{\alpha_2}{\alpha_1}, V_3 \geq V_1, V_3 \geq V_2$ or $p_0 \leq I < \frac{ap_1\alpha_2}{\alpha_1}, V_1 \leq V_3 \ \Rightarrow$ *solution* 3
- $I \geq \max(p_0, \frac{ap_1\alpha_2}{\alpha_1}), I < p_0 + p_1(a+1)\frac{\alpha_2}{\alpha_1}, V_3 < V_2$ *or* $\frac{ap_1\alpha_2}{\alpha_1} \leq I < p_0$ *or* $I \geq p_0 + p_1(a+1)\frac{\alpha_2}{\alpha_1}, p_1 < p_0$ $\Rightarrow$ *solution* 2
- $I \geq p_0 + p_1(a+1)\frac{\alpha_2}{\alpha_1}, p_1 \geq p_0 \Rightarrow$ *solution* 4

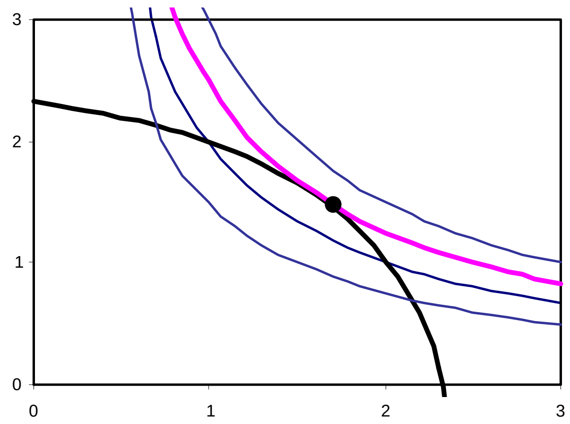
Problem 3

(a) $\max\{x_1x_2/(x_1+x_2)|x_1p_1+x_2p_2 \le I, x_i \ge 0\}$
$\pounds = x_1x_2/(x_1+x_2) + \lambda(I - x_1p_1 - x_2p_2)$
First order conditions:
$\frac{\partial \pounds}{\partial x_i} = (x_{-i}/(x_1+x_2))^2 - \lambda p_i = 0 \ \cap \ x_1p_1 + x_2p_2 = I$
The solution is $x_{-i}/\sqrt{p_i} = (x_1+x_2)\sqrt{\lambda} \Rightarrow x_1\sqrt{p_1} = x_2\sqrt{p_2}$
Demand functions: $x_i(p,I) = \frac{\sqrt{p_i}}{\Sigma\sqrt{p_j}}\frac{I}{p_i}$, for all $i = 1,2$
Maximized utility: $v(p,I) = \frac{I}{(\sqrt{p_1}+\sqrt{p_2})^2}$

(b) $\max\{x_3x_1x_2/(x_1+x_2)|x_1p_1+x_2p_2+x_3p_3 \le I, x_i \ge 0\} =$
$\max\{\frac{x_3 y}{(\sqrt{p_1}+\sqrt{p_2})^2}|x_3p_3 + y \le I, x_i \ge 0\}$
That is Cobb-Douglas: $x_3(p,I) = \frac{I}{2p_3}, y = \frac{I}{2} \Rightarrow$
$x_i(p,I) = \frac{\sqrt{p_i}}{\sqrt{p_1}+\sqrt{p_2}}\frac{I}{2p_i}$, for $i = 1,2$

Problem 4

(a) This is the definition of PE allocation:
$\max\{\alpha_1\ln(a_1+x_1) + \alpha_2\ln(a_2+x_2)|x_1+y_1 \le w_1, x_2+y_2 \le w_2,$
$\alpha_1\ln(b_1+y_1) + \alpha_2\ln(b_2+y_2) \ge U_2, x_i \ge 0, y_i \ge 0\}$
It means that given some indifference curve of the second guy we search for a point on it, that gives the other guy maximum utility. At that point the indifference curves should be tangent if it is in the interior of the Edgeworth box.



The definition simplifies to for the interior cases:
$\max\{\alpha_1\ln(a_1+w_1-y_1) + \alpha_2\ln(a_2+w_2-y_2)|$
$\alpha_1\ln(b_1+y_1) + \alpha_2\ln(b_2+y_2) \ge U_2, y_i \ge 0\}$
Lagrangian: $\pounds = \alpha_1\ln(a_1+w_1-y_1) + \alpha_2\ln(a_2+w_2-y_2) +$

$$\lambda(\alpha_1 \ln(b_1 + y_1) + \alpha_2 \ln(b_2 + y_2) - U_2)$$

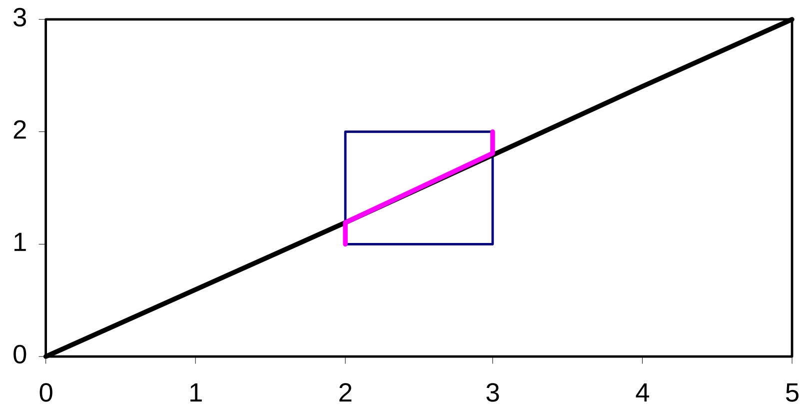
FOC for interior: $\lambda = \frac{y_1+b_1}{a_1+w_1-y_1} = \frac{y_2+b_2}{a_2+w_2-y_2} \Rightarrow$

$y_1(a_2 + w_2 + b_1) + b_1(a_2 + w_2) = y_2(a_1 + w_1 + b_2) + b_2(a_1 + w_1)$

(b) For the case $a_1 = b_1 = 0$ we get: $y_1 w_2 = y_2 w_1$
This is the diagonal of the Edgeworth box.

(c) The condition derived from FOC is a line.

(d) Redefining $z_i = b_i + y_i$ gives: $\frac{z_1}{a_1+w_1+b_1-z_1} = \frac{z_2}{a_2+w_2+b_2-z_2} \Rightarrow$
$z_1(a_2 + w_2 + b_2) = z_2(a_1 + w_1 + b_1)$
That means our Ebox is a subset of a bigger Ebox
in z_i coordinates in which PE allocations are on the diagonal.
For $a = (2; 1)$ $b = (2; 1)$ $w = (1; 1)$ the picture looks like this:



(e) The example values are: $a = (1; 4)$ $b = (4; 1)$ $w = (1; 1)$.