TA section, Week 5, Handout Anton Cheremukhin

April 30, 2009

Consumption and Savings

Robinson Crusoe chooses how many coconuts C_1 to consume this year and how many to consume next year C_2. This year's crop is Y_1 and next year's crop is Y_2. Robinson values coconuts equally over the years. If Robinson saves coconuts, he gets a gross return $R = 1 + r$. Robinson's preferences are defined as follows:

The first resource constraint says that all the coconuts from this year are either consumed or saved:

$$C_1 + S = Y_1$$

The second resource constraint says that all the coconuts from next year's crop plus savings from the first year are consumed:

$$C_2 = S\,(1 + r) + Y_2$$

Robinson maximizes utility given the constraints:

$$U\,(C_1, C_2) = \ln C_1 + \ln C_2$$

We want to find how many coconuts he will consume this year, how many will be saved, and how many he will consume next year. Solution:

First derive the lifetime budget constraint:

$$S_1 = \tfrac{C_2 - Y_2}{1+r} \qquad \Rightarrow \qquad Y_1 = C_1 + S = C_1 + \tfrac{C_2 - Y_2}{1+r}$$
$$\Rightarrow \qquad C_1 + \tfrac{C_2}{1+r} = Y_1 + \tfrac{Y_2}{1+r}$$

This equation says that the present value of lifetime consumption is equal to the present value of lifetime income.

Robinson's problem is therefore:

$$\ln C_1 + \ln C_2 \to \max_{C_1, C_2}$$
$$\text{s.t.} \qquad C_1 + \tfrac{C_2}{1+r} = Y_1 + \tfrac{Y_2}{1+r} = I$$

Substitute:

$$\ln\left(I - \tfrac{C_2}{1+r}\right) + \ln C_2 \to \max_{C_2}$$
$$\text{FOC:} \qquad \frac{\frac{1}{1+r}}{I - \frac{C_2}{1+r}} = \frac{1}{C_2}$$

Therefore, $\frac{C_2}{1+r} = I - \frac{C_2}{1+r}$

Hence,

$$C_2 = \frac{I(1+r)}{2} = \frac{Y_1(1+r)+Y_2}{2}$$

$$C_1 = I - \frac{C_2}{1+r} = \frac{Y_1(1+r)+Y_2}{2(1+r)}$$

$$S = Y_1 - C_1 = \frac{Y_1(1+r)-Y_2}{2(1+r)}$$

Let $Y_1 = 0, Y_2 = 500, r = 10\%$

Then

$$C_2 = \frac{0(1+0.1)+500}{2} = 250$$

$$C_1 = \frac{0(1+0.1)+500}{2(1+0.1)} = 227.3$$

$$S = \frac{0(1+1.1)-500}{2(1+0.1)} = -227.3$$

Let now preferences be defined as

$$U(C_1, C_2) = \frac{C_1^\gamma}{\gamma} + \frac{C_2^\gamma}{\gamma}$$

Then Robinson's problem boils down to:

$$\frac{1}{\gamma}\left(I - \frac{C_2}{1+r}\right)^\gamma + \frac{1}{\gamma}(C_2)^\gamma \to \max_{C_2}$$

$$\text{FOC:} \quad \frac{1}{1+r}\left(I - \frac{C_2}{1+r}\right)^{\gamma-1} = (C_2)^{\gamma-1}$$

Therefore, $C_2 (1+r)^{\frac{1}{\gamma-1}} = I - \frac{C_2}{1+r}$

Hence,

$$C_2 = \frac{I(1+r)}{1+(1+r)^{\frac{\gamma}{\gamma-1}}} = \frac{Y_1(1+r)+Y_2}{1+(1+r)^{\frac{\gamma}{\gamma-1}}}$$

$$C_1 = I - \frac{C_2}{1+r} = \frac{Y_1(1+r)+Y_2}{1+(1+r)^{\frac{\gamma}{\gamma-1}}} \frac{(1+r)^{\frac{\gamma}{\gamma-1}}}{1+r}$$

$$S = Y_1 - C_1$$

Let $Y_1 = 50, Y_2 = 200, r = 5\%$

Case 1: $\gamma = -0.05$

Then

$$C_2 = \left.\frac{50(1+0.05)+200}{1+(1+0.05)^{\frac{\gamma}{\gamma-1}}}\right|_{\gamma=-0.05} = 126.1$$

$$C_1 = \left.\frac{50(1+0.05)+200}{1+(1+0.05)^{\frac{\gamma}{\gamma-1}}} \frac{(1+0.05)^{\frac{\gamma}{\gamma-1}}}{1+0.05}\right|_{\gamma=-0.05} = 120.38$$

$$S = Y_1 - C_1 = 50 - 120.38 = -70.38$$

Case 1: $\gamma = -20$

Then

$$C_2 = \left.\frac{50(1+0.05)+200}{1+(1+0.05)^{\frac{\gamma}{\gamma-1}}}\right|_{\gamma=-20} = 123.32$$

$$C_1 = \left.\frac{50(1+0.05)+200}{1+(1+0.05)^{\frac{\gamma}{\gamma-1}}} \frac{(1+0.05)^{\frac{\gamma}{\gamma-1}}}{1+0.05}\right|_{\gamma=-20} = 123.03$$

$$S = Y_1 - C_1 = 50 - 123.03 = -73.03$$

General Equilibrium

Robinson Crusoe lives on an island alone, so he acts both as a firm and as a household.

As a firm he produces coconuts according to a technology:

$$Y = 9L\left(1 - L\right),$$

where L is the amount of labor hired. The Robinson-the-firm hires Robinson-the-household to work and pays profits back to Robinson-the-household.

The problem of Robinson-the-firm is to maximize profits:

$$\begin{gathered}\pi = Y - wL = 9L\left(1 - L\right) - wL \to \max_L \\ \text{FOC:} \qquad 9\left(1 - 2L\right) - w = 0\end{gathered}$$

Thus, labor demand is determined by:

$$L^d = \tfrac{1}{2}\left(1 - \tfrac{w}{9}\right)$$

Profits in equilibrium are:

$$\begin{gathered}\pi = 9\left(L - L^2\right) - wL = 9\left(L - 2L^2\right) - wL + 9L^2 = \\ = L\left[9\left(1 - 2L\right) - w\right] + 9L^2 = 9L^2\end{gathered}$$

As a household Robinson earns a wage by working at the firm, get the profits and spends all of his income to buy coconuts. He likes coconuts but dislikes working, so he solves the following problem:

$$\begin{gathered}U\left(C, L\right) = \ln C + \ln\left(1 - L\right) \to \max_{C,L} \\ \text{s.t.} \qquad C = wL + \pi\end{gathered}$$

Substitute:

$$\begin{gathered}\ln\left(wL + \pi\right) + \ln\left(1 - L\right) \to \max_L \\ \text{FOC:} \qquad \tfrac{w}{wL+\pi} = \tfrac{1}{1-L} \\ w - wL = wL + \pi \\ 2wL = w - \pi \\ w\left(1 - 2L\right) = \pi\end{gathered}$$

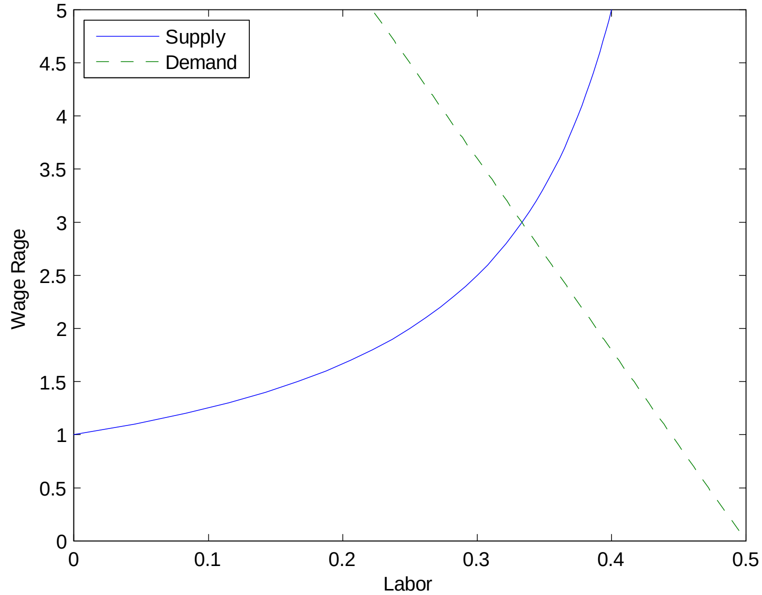
Labor supply is then determined by:

$$L^s = \tfrac{1}{2}\left(1 - \tfrac{\pi}{w}\right)$$

In equilibrium labor demand is equal labor supply:

$$L^s = \tfrac{1}{2}\left(1 - \tfrac{\pi}{w}\right) = L^d = \tfrac{1}{2}\left(1 - \tfrac{w}{9}\right)$$

This is depicted in a graph:



Equivalently the real wage w equalizes demand and supply:

$$w = 9\left(1 - 2L^d\right) = w = \frac{\pi}{1-2L^s}$$

Remember, that in equilibrium $\pi = 9L^2$
Hence, in equilibrium: $9\left(1 - 2L\right) = \frac{9L^2}{1-2L}$
Thus, $(1 - 2L)^2 = L^2$
Therefore, $1 - 2L = L \quad \Rightarrow \quad L = \frac{1}{3}$
In equilibrium,

$$L^s = L^d = \frac{1}{3},$$
$$w = 9\left(1 - 2\frac{1}{3}\right) = 3,$$
$$\pi = 9\frac{1}{3^2} = 1,$$
$$C = wL + \pi = 1 + 1 = 2,$$
$$Y = 9\frac{1}{3}\left(1 - \frac{1}{3}\right) = 2$$