Econ202B Review Session for Midterm Anton Cheremukhin

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Exercise 1 *Business cycle volatility and runup in stock prices (Comp Fall 03, Question 3)*

Consider a representative agent economy. The RA has separable CRRA utility. There is a single Lucas tree that pays off the aggregate endowment every period. The growth rate of the aggregate endowment $\Delta \ln C_t$ is i.i.d. and normally distributed with mean μ and standard deviation σ.

(.) State the sequential and recursive problems

(i) Compute the price-dividend ratio in closed form.

(ii) Compute the riskless rate in closed form.

(iii) Some people have argued that the recent runup in stock prices is due to a drop in business cycle volatility. In the model, what happens to prices and interest rates when σ falls? Interpret the result. Does the argument make sense?

Solution.

SeqCE: Allocations $\{c_t, s_{t+1}, b_{t+1}\}$ and Prices $\{1, q_t, p_t\}$ s.t.

(1) Allocation solves: $\max_{c_t, s_{t+1}, b_{t+1}} \left\{ E \sum_{t=0}^{\infty} \beta^t u(c_t) \,|\, b_t + s_t (q_t + d_t) \geq c_t + q_t s_{t+1} + p_t b_{t+1} \right\}$ given price.

(2) Prices satisfy Market Clearing (corresponding price):

Consumption $c_t = d_t | (1)$ Risky equity: $s_t = 1 | (q_t)$ Riskless bond: $b_t = 0 | (p_t)$

(3) Dividends d_t are exogenous and their distribution is known.

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Bellman: $V(s, b, S, B, x) = \max_{s', b'} \left[u(b + s(q + d(x)) - qs' - pb') + \beta E_{x'|x} V(s', b', S', B', x') \right]$

RCE: (1) Value function: $V(s, b, S, B, x)$

(2) Decision Rules $s'(s, b, S, B, x), b'(s, b, S, B, x)$

(3) Price Functions $q(s, b, S, B, x), p(s, b, S, B, x)$

(4) Laws of Motion of aggregate holdings $H_s(S, B, x), H_k(S, B, x)$.

s.t.

(i)Decision Rules solve Bellman given Price Functions and Laws of Motion.

(ii)Price Functions satisfy market clearing given the state (S, B, x): $s' = S' = 1, \quad b = B' = 0.$

(iii)Perceptions are correct: $H_s(S, B, x) = s'(S, B, S, B, x)$, $H_b(S, B, x) = b'(S, B, S, B, x)$

Characterization of equilibrium:

FOC: $qu'_c = \beta E_{x'|x} V'_s(s', b', S', B', x')$

$pu'_c = \beta E_{x'|x} V'_b(s', b', S', B', x')$

Envelope: $V'_s(s, b, S, B, x) = (q + d(x)) u'_c$

$V'_b(s, b, S, B, x) = u'_c$

Euler equations follow:

$$qu'_c = \beta E_{x'|x} \left[(q' + d(x')) u'_{c'} \right] \qquad pu'_c = \beta E_{x'|x} \left[u'_{c'} \right]$$

Asset prices therefore are determined by:

$q = \beta E_{x'|x} \left[\frac{u'_{c'}}{u'_c} (q' + d(x')) \right] \qquad p = \beta E_{x'|x} \left[\frac{u'_{c'}}{u'_c} \right]$

Now use the fact that $u(c) = \frac{d^{1-\rho}}{1-\rho}$. Then $\frac{u'_{c'}}{u'_c} = \left(\frac{c}{c'} \right)^{\rho}$. Also, remember, that $c = d$. Hence,

$p_t = \beta E_t \left[\left(\frac{c_t}{c_{t+1}} \right)^{\rho} \right] \qquad q_t = \beta E_t \left[\left(\frac{c_t}{c_{t+1}} \right)^{\rho} (q_{t+1} + c_{t+1}) \right]$

These equations we can solve recursively for the equity price:

$q_t = \beta E_t \left[\left(\frac{c_t}{c_{t+1}} \right)^{\rho} (q_{t+1} + c_{t+1}) \right] = \beta E_t \left[\left(\frac{c_t}{c_{t+1}} \right)^{\rho} \left(\beta \left(\frac{c_{t+1}}{c_{t+2}} \right)^{\rho} (q_{t+2} + c_{t+2}) + c_{t+1} \right) \right] =$
$= E_t \left[\beta \left(\frac{c_t}{c_{t+1}} \right)^{\rho} c_{t+1} + \beta^2 \left(\frac{c_t}{c_{t+2}} \right)^{\rho} (q_{t+2} + c_{t+2}) \right] = ...$

Using the transversality conditions we take the limit: $\lim_{j \to \infty} \beta^j \left(\frac{c_t}{c_{t+j}} \right)^{\rho} q_{t+j} = 0$.

Then the **closed forms** for the prices are: $p_t = E_t \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\rho}$.

$q_t = E_t \sum_{j=1}^{\infty} \beta^j c_t^{\rho} c_{t+j}^{1-\rho} = E_t \sum_{j=1}^{\infty} \beta^j c_t \left(\frac{c_{t+j}}{c_t} \right)^{1-\rho} = E_t \sum_{j=1}^{\infty} \beta^j c_t \left(\left(\frac{c_{t+1}}{c_t} \right) \left(\frac{c_{t+2}}{c_{t+1}} \right) ... \left(\frac{c_{t+j}}{c_{t+j-1}} \right) \right)^{1-\rho}$

Now use the fact that $\Delta \ln c_{t+1} = \ln \frac{c_{t+1}}{c_t} = \varepsilon_{t+1} \sim N(\mu, \sigma^2)$

Hence, $\ln \left(\frac{c_{t+1}}{c_t} \right)^{-\rho} = -\rho \varepsilon_{t+1} \sim N(-\rho\mu, \rho^2\sigma^2)$ i.i.d.

$\ln \left(\frac{c_{t+1}}{c_t} \right)^{1-\rho} = (1-\rho) \varepsilon_{t+1} \sim N\left((1-\rho)\mu, (1-\rho)^2 \sigma^2 \right)$ i.i.d.

$\ln \left(\left(\frac{c_{t+1}}{c_t} \right) \left(\frac{c_{t+2}}{c_{t+1}} \right) ... \left(\frac{c_{t+j}}{c_{t+j-1}} \right) \right)^{1-\rho} = (1-\rho)(\varepsilon_{t+1} + \varepsilon_{t+2} + ... + \varepsilon_{t+j}) \sim N\left((1-\rho) j\mu, (1-\rho)^2 j\sigma^2 \right)$

Using the fact that, $E \exp u_t = \exp \left(E[u_t] + \frac{Var[u_t]}{2} \right)$, denote $\exp \left((1-\rho)\mu + \frac{(1-\rho)^2\sigma^2}{2} \right) = \varphi$

$q_t = \sum_{j=1}^{\infty} \beta^j c_t \exp \left((1-\rho) j\mu + \frac{(1-\rho)^2 j\sigma^2}{2} \right) = c_t \sum_{j=1}^{\infty} (\beta\varphi)^j = c_t \frac{\beta\varphi}{1-\beta\varphi}$

Dividend-price ratio: $\boxed{c_t/q_t = \frac{1}{\beta \exp\left((1-\rho)\mu + \frac{(1-\rho)^2\sigma^2}{2} \right)} - 1}$

$p_t = \beta E_t \left(\frac{c_{t+1}}{c_t} \right)^{-\rho} = \beta \exp \left(-\rho\mu + \frac{\rho^2\sigma^2}{2} \right) = \frac{1}{1+r}$

Risk-free rate: $\boxed{r_t = \frac{1}{\beta} \exp \left(\rho\mu - \frac{\rho^2\sigma^2}{2} \right) - 1}$

$\frac{\partial}{\partial\sigma}(r_t) = -\rho^2 \frac{\sigma}{\beta} \exp \left(\rho\mu - \frac{\rho^2\sigma^2}{2} \right) < 0 \qquad \frac{\partial}{\partial\sigma}(c_t/q_t) = -\frac{1}{\beta} \frac{(1-\rho)^2}{\exp\left((1-\rho)\mu + \frac{(1-\rho)^2\sigma^2}{2} \right)} \sigma < 0$

These results imply that when volatility falls, the risk free rate grows and the stock prices fall. This contradicts the statement that stock prices have been growing because the volatility decreased.

Exercise 2 *Comp Spring 06, Question 4*

Consider an economy extending over two periods t=1,2 which is populated by a continuum of ex-ante identical households of measure one, indexed by i. Preferences are given by:

$E\left[\sqrt{c_{i,1}(1-n_{i,1})} + \beta\sqrt{c_{i,2}(1-n_{i,2})} \right]$. *where $c_{i,t}$ is consumption and $n_{i,t}$ is labor. During period 1 which corresponds to education people enter one of two professions, business or economics. Business does not require education hence $n_{i,1} = 0$ for a businessperson, while economics requires a Ph.D., hence $n_{i,1} = 0.75$ for an economist. During the second period members of both occupations work $n_{i,2} = 0.5$. During the first period consumption is derived from an aggregate endowment of $R = 1$. The consumption good cannot be stored. During the second period the consumption good is produced using a production function $F(B,E) = \sqrt{BE}$ where B is the mass of businesspeople and E is the mass of economists. Given that everybody has to choose an occupation B+E=1.*

a) Specify the economy in the language of Debreu. Make sure to use a lottery to deal with the indivisible occupational choice (no part time).

b) In equilibrium are there going to be more economists or more businesspeople? Which group will have higher consumption in period 1, in period 2?

Solution:

Consumers will choose over probabilities of becoming a business person (π) or an economist $(1-\pi)$, and over consumptions in each of the periods in each case. Given the probabilities, there is no choice left of how much to work.

Therefore the commodity space is 5-dimensional: $S = R^5 = \{c_{B1}, c_{B2}, c_{E1}, c_{E2}, \pi_B\}$.

$X = \{x \in S | x_{1,2,3,4} \geq 0, 0 \leq x_5 \leq 1\}$

$U(x) = x_5 \left[\sqrt{x_1(1-0)} + \beta\sqrt{x_2(1-0.5)}\right] + (1-x_5)\left[\sqrt{x_3(1-0.75)} + \beta\sqrt{x_4(1-0.5)}\right]$

If there are y_5 business people and $(1-y_5)$ economists, then the total production is $\sqrt{y_5(1-y_5)}$:

$Y = \left\{y \in S | y_j \geq 0, 0 \leq y_5 y_1 + (1-y_5) y_3 \leq 1, 0 \leq y_5 y_2 + (1-y_5) y_4 \leq \sqrt{y_5(1-y_5)}, 0 \leq y_5 \leq 1\right\}$

Resource constraint: $x = y$.

Solution of the planner's problem can be simplified to get:

$y_5\left[\sqrt{y_1} + \beta\sqrt{0.5y_2}\right] + (1-y_5)\left[\sqrt{0.25y_3} + \beta\sqrt{0.5y_4}\right] \to \max$

$s.t. \qquad (\lambda) \qquad y_5 y_1 + (1-y_5) y_3 = 1, \qquad y_5 y_2 + (1-y_5) y_4 = \sqrt{y_5(1-y_5)} \qquad (\mu)$

FOC: $\quad \frac{1}{2\sqrt{y_1}} = \lambda = \frac{\sqrt{0.25}}{2\sqrt{y_3}} \qquad \frac{\beta\sqrt{0.5}}{2\sqrt{y_2}} = \mu = \frac{\beta\sqrt{0.5}}{2\sqrt{y_4}}$

$\left[\sqrt{y_1} + \beta\sqrt{0.5y_2}\right] - \left[\sqrt{0.25y_3} + \beta\sqrt{0.5y_4}\right] - \lambda y_1 + \lambda y_3 - \mu y_2 + \mu y_4 + \mu\frac{1-2y_5}{2\sqrt{y_5(1-y_5)}} = 0$

Therefore, $\frac{\sqrt{y_3}}{\sqrt{y_1}} = \sqrt{0.25} \qquad \Rightarrow \qquad y_1 = 4y_3, \qquad y_2 = y_4$

Consumption of business people is four times higher then that of economists in period 1 and the same in period 2.

$\left[\sqrt{y_1} + \beta\sqrt{0.5y_2}\right] - \left[\sqrt{0.25y_3} + \beta\sqrt{0.5y_4}\right] - \frac{\sqrt{y_1}}{2} + \frac{\sqrt{0.25y_3}}{2} - \frac{\beta\sqrt{0.5y_2}}{2} + \frac{\beta\sqrt{0.5y_4}}{2} + \frac{\beta\sqrt{0.5}}{2\sqrt{y_4}}\frac{1-2y_5}{2\sqrt{y_5(1-y_5)}} = 0$

$\left[\sqrt{4y_3} + \beta\sqrt{0.5y_4}\right] - \left[\sqrt{0.25y_3} + \beta\sqrt{0.5y_4}\right] - \frac{\sqrt{4y_3}}{2} + \frac{\sqrt{0.25y_3}}{2} - \frac{\beta\sqrt{0.5y_4}}{2} + \frac{\beta\sqrt{0.5y_4}}{2} + \frac{\beta\sqrt{0.5}}{2\sqrt{y_4}}\frac{1-2y_5}{2\sqrt{y_5(1-y_5)}} = 0$

$\frac{3}{4}\sqrt{y_3} + \frac{\beta\sqrt{0.5}}{2\sqrt{y_4}}\frac{1-2y_5}{2\sqrt{y_5(1-y_5)}} = 0 \qquad RC: \qquad (3y_5+1) y_3 = 1, \qquad y_4 = \sqrt{y_5(1-y_5)}$

$\sqrt{\frac{18}{3y_5+1}} = \frac{\beta}{\sqrt{\sqrt{y_5(1-y_5)}}}\frac{2y_5-1}{\sqrt{y_5(1-y_5)}} \qquad \Rightarrow \qquad y_5 \geq \frac{1}{2}$

There are more business people then economists.

$(y(1-y))^{\frac{3}{4}}\left(\frac{18}{3y+1}\right)^{\frac{1}{2}} = \beta(2y-1)$ has a solution for any $\beta \in [0,1]$

For example: $\quad \{\beta = 0 \to y = 1\} \qquad \left\{\beta = \frac{1}{3} \to y = 0.928\,37\right\} \qquad \left\{\beta = \frac{1}{2} \to y = 0.889\,31\right\}$
$\left\{\beta = \frac{2}{3} \to y = 0.853\,91\right\} \qquad \left\{\beta = \frac{4}{5} \to y = 0.828\,59\right\} \qquad \{\beta = 1 \to y = 0.795\,39\}$

As β goes to infinity, y approaches $\frac{1}{2}$.